Exceptional collections and D-branes probing toric singularities

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ABSTRACT: We demonstrate that a strongly exceptional collection on a singular toric surface can be used to derive the gauge theory on a stack of D3-branes probing the Calabi-Yau singularity caused by the surface shrinking to zero size. A strongly exceptional collection, i.e., an ordered set of sheaves satisfying special mapping properties, gives a convenient basis of D-branes. We find such collections and analyze the gauge theories for weighted projective spaces, and many of the $Y^{p,q}$ and $L^{p,q,r}$ spaces. In particular, we prove the strong exceptionality for all p in the $Y^{p,p-1}$ case, and similarly for the $Y^{p,p-2r}$ case.

KEYWORDS: Differential and Algebraic Geometry, D-branes, AdS-CFT Correspondence.

Contents

1. Introduction

Determining the low-energy gauge theory on a stack of D-branes probing a Calabi-Yau singularity is an important, interesting, and in general unsolved problem. These D-brane constructions can be used to build flux vacua in string theory, and they play an important role in the AdS/CFT correspondence, where they yield a geometric understanding of strongly coupled gauge theories. While many simple singularities, such as the conifold and orbifolds (see for example [1] and [2]), have been successfully analyzed, the general case remains elusive.

We analyze D-branes probing the singularity of toric Calabi-Yau three-folds. More precisely, our spaces are local Calabi-Yau varieties, and can be represented as the total space of the canonical sheaf of a singular compact complex surface. While not all Calabi-Yau singularities are toric, the toric subset provides a vast number of interesting examples which, because of their three U(1) isometries, are easier to understand. The conifold, $\mathbb{C}^3/\mathbb{Z}_n$ and $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$, where the orbifold action preserves the SU(3) holonomy, are all examples of toric Calabi-Yau singularities.

This paper in large part is motivated by the discovery of metrics on an infinite class of toric Calabi-Yau three-fold cones [3, 4] — the $Y^{p,q}$ and $L^{p,q,r}$ spaces — and subsequent developments. Let Y denote the five dimensional level surface of the cone X. This class of metrics describes all toric Calabi-Yau singularities where the topology of Y is $S^2 \times S^3$. Previously only the metrics for the conifold and $\mathbb{C}^3$ were known! The new explicit metrics allow for independent confirmation of the results that we obtain using more abstract algebro-geometric techniques.

We study these Calabi-Yau singularities using exceptional collections [5]. Exceptional collections are in essence a "nice" basis of D-branes for describing the gauge theory. More precisely, an exceptional collection is an ordered set of coherent sheaves satisfying special requirements regarding the maps between the sheaves. These collections do not actually live on the Calabi-Yau cone but rather on the base of the cone. Here again the Calabi-Yau space is viewed as the canonical sheaf K_V over a given compact surface V.

Exceptional collections have appeared in the physics literature before. Applied first to Landau Ginzburg models [7, 8, 6], they more recently have reappeared in the AdS/CFT literature [11, 10]. Their connection with monodromy was elucidated in [9].

Exceptional collections are a useful tool for studying CY singularities for a number of reasons:

1. Although we study only toric examples in this paper, exceptional collections can be used to study non-toric singularities as well, as pointed out in the AdS/CFT context in [11]. For example, collections exist for all del Pezzo singularities, only some of which are toric [13, 10, 12]. Hopefully, insights gained here can be applied to the general case.

2. Exceptional collections provide a rigorous and constructive way of converting geometry into gauge theory. Having constructed the gauge theory from the exceptional collection, one can be *absolutely sure* the gauge theory is the low energy effective description of the D-branes probing the singularity.

3. These collections allow one to understand all the different Seiberg dual phases of the gauge theory. To each singularity, one can typically associate many gauge theories, all related to each other by Seiberg duality [14]. In the exceptional collection language, Seiberg duality is realized as a special sequence of mutations. On the other hand, methods adapted to toric geometry allow one to access only the "toric" phases, where the ranks of the gauge groups are all equal [15, 16].

4. Algorithms that convert an exceptional collection into a quiver can be implemented on a computer and are typically quite fast. With the computer technology available today, these computations generically take seconds to run. Even the more complicated examples take only a couple of minutes.

This paper is first and foremost a proof of principle that exceptional collections exist and are useful for studying singularities where a singular two dimensional complex surface V shrinks to zero size. More specifically, the singular Calabi-Yau variety X arises as a particular limit point in the Kähler moduli space of the canonical bundle over V, where V itself undergoes a generalized Type II divisorial contraction.

Previously in the physics literature exceptional collections were used only for smooth surfaces V, which essentially meant that V was del Pezzo. All of our interesting examples take V to be a singular toric surface. We will find in some cases more convenient to think of V as a stack, rather than just as a space. In these cases, we need the stackiness in order to keep track of the extra information about how V is embedded in the cone.[1] Although V can only have orbifold singularities,[2] shrinking V typically produces more general types of singularities in the Calabi-Yau than orbifolds.

In deriving the quiver from the singularity, the first step is to discover an exceptional collection. While we do not have a general algorithm for producing the collection, we do have several useful tools, which we can combine:

- We know how to write down an appropriate collection for any weighted projective space $\mathbb{P}^2(a, b, c)$.

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[1] Although V is singular, the cone will generically be smooth everywhere except at the "tip".

[2] Toric varieties are always normal, and hence, in two dimensions, the singularities are isolated, and in fact quotient.

- Given a surface and a collection on it, we can generate a collection on the new surface obtained by blowing up a point. This procedure is particularly useful when the point in question is smooth.

- Given a surface which is a direct product of two curves, taking the exterior tensor product of the collections on the curves, and ordering it appropriately, gives a collection on the product space.

Perhaps the most important tool in discovering an exceptional collection is being able to check that it is indeed exceptional. Taking advantage of some recent mathematical results in local cohomology [17], we have implemented a routine in Macaulay2 which checks whether or not a given collection is exceptional.

The Macaulay2 routine, as any computer algorithm, only works on one example at a time. But one would like to treat an entire class of collections all at once. Fortunately, after studying several examples by Macaulay2, we get a glimpse of the general structure in several cases. Then, using algebraic-geometric tools, we are able to prove that our collections on $Y^{p,p-1}$, $Y^{p,p-2}$, and more generally $Y^{p,q}$, where $p-q$ is even,[3] are exceptional for all p and q.

To convert the collection into a quiver gauge theory, once we know it is exceptional, we need to calculate all the maps between the sheaves. Usually this calculation is impossible to do by hand. However, in our Macaulay2 routine the maps are a byproduct. We have an even faster Maple routine, which computes the same maps, but cannot always deal with the Ext^1's.

The two routines are based on very different mathematics. The Maple routine counts lattice points inside polytopes, which is a linear programming problem. The Macaulay2 routine on the other hand uses a purely algebraic *local cohomology* calculation. Since this approach is completely new in the physics literature, we include a more detailed description of it. The fact that the two procedures *always* gave us the same answer for the Ext^0's and Ext^2's provides us with a strong consistency check.

In the next section, we introduce the mathematical machinery needed to write down collections of sheaves on these toric surfaces and check their exceptionality. Then in sections 3, 4 and 5 we derive exceptional collections for a dozen or so specific examples and from these collections calculate the gauge theory. As a warm-up exercise, we rederive collections for the toric del Pezzos, but now using toric techniques. We also derive collections for some simple $\mathbb{C}^3/\mathbb{Z}_n$ orbifolds, finding familiar quivers. Finally, we look at the more complicated and interesting $L^{p,q,r}$ singularities of which the $Y^{p,q}$'s are a subset. (We also consider the $X^{p,q}$ [18].) Section 6 contains mathematical proofs that our collections for $Y^{p,p-1}$ and $Y^{p,p-2r}$ are exceptional.

In the last section of the paper, in an inversion of the philosophy espoused above, we compute a quiver for the $L^{p,q,r}$ directly from the toric data. We then give an algorithm for converting the quiver into a collection we believe should in general be strongly exceptional.

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[3]There is another technical condition on p and q, as discussed in section 6.

2. The specifics

2.1 From cones to surface

Our goal is to develop an algorithm for converting a toric Calabi-Yau cone into a quiver gauge theory via exceptional collections. We take steps in this direction, by providing methods for generating strongly exceptional collections of line bundles from the toric diagram, and then a procedure for converting that collection into a quiver gauge theory.

Our starting point is the toric data for a local Calabi-Yau three-fold. In other words, we are given a *coplanar* collection of at least three vectors in $\mathbb{Z}^3$. For example, $V_1 = (1, 1, 0)$, $V_2 = (1, 0, 1)$ and $V_3 = (1, -a, -b)$, with a and b positive integers, describe an $\mathcal{N} = 1$ supersymmetry preserving orbifold $\mathbb{C}^3/\mathbb{Z}_{1+a+b}$. These three vectors V_i indeed end on the plane $x = 1$.

The next step is to convert this three dimensional cone into a two dimensional fan describing a compact, possibly singular toric surface. Assume all the vectors for the $3d$ cone are written in the form $V_i = (1, *, *)$. Truncating the first entry of the vector, we obtain a set of two dimensional vectors $\{v_i\}$. These v_i define the vertices of a polygon. If this polygon contains one or more internal lattice points, then we can partially resolve the $3d$ singularity by blowing up a four cycle.

It is in this conversion of cone to surface that we are naturally led into the language of stacks. The v_i that arise in this procedure are often not primitive, i.e. there is some positive integer $n > 1$ such that v_i/n is also a lattice point. These types of fans are simple, two dimensional examples of the toric stacks described in [19].

The choice of the internal lattice point determines the origin of the two dimensional fan. For example, for $\mathbb{C}^3/\mathbb{Z}_{1+a+b}$ and choosing the point $(1, 0, 0)$ as the origin of the fan, we get the new set of vectors $v_1 = (1, 0)$, $v_2 = (0, 1)$, and $v_3 = (-a, -b)$ which define the weighted projective space $\mathbb{P}(1, a, b)$. For large a and b, there are generically many interior lattice points and hence many choices of origin. The gauge theory should be independent of our choice, and in practice we choose the lattice point that gives the simplest surface. (We will see this independence in some of the examples.) Sometimes there is no lattice point inside the polygon — for example for the conifold — in which case this exceptional collection method needs modification.[4]

The fact that the fan descends from a cone means that the fan must define the corners of a convex polygon. For example, consider the Hirzebruch surfaces $\mathbb{F}_n$ which are described by the four vectors $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, n)$, and $v_4 = (0, -1)$. For $n = 0$ or $n = 1$, these four vectors define a convex quadrilateral from which we can construct Calabi-Yau cones. However for $n > 1$, the vector v_2 lies inside the triangle defined by v_1, v_3 and v_4 and the Calabi-Yau cone is better thought of as a $\mathbb{C}^3/\mathbb{Z}_{2+n}$ orbifold.

Starting with a Calabi-Yau cone, our problem can be rephrased in terms of finding an exceptional collection for a special kind of compact toric surface (or really toric stack).

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[4]In the case where there is no surface to blow up, we expect to be able to blow up a curve and write an exceptional collection for the curve.

The surface X is described by a fan of at least three vectors in $\mathbb{Z}^2$, and in our case the vectors describe a convex polygon. The vectors do not need to be primitive.

2.2 Exceptional collections

An **exceptional collection** of sheaves $\mathcal{E} = (E_1, E_2, \ldots, E_n)$ is an ordered set of sheaves which satisfy the following special properties:

1. Each E_i is exceptional: $\mathrm{Ext}^q(E_i, E_i) = 0$ for $q > 0$ and $\mathrm{Ext}^0(E_i, E_i) = \mathrm{Hom}(E_i, E_i) = \mathbb{C}$.

2. $\mathrm{Ext}^q(E_i, E_j) = 0$ for $i > j$ and $\forall q$.

In these notes, we will be most interested in the case where the collection is **strongly exceptional**, in which case $\mathrm{Ext}^q(E_i, E_j) = 0$ for $i < j$ and $q > 0$. For smooth toric surfaces, the collection must be strong to generate a physical quiver gauge theory [20, 21], and the same is true for singular surfaces as well.

For the most part, our sheaves can be thought of as line bundles, and line bundles are easy to describe in a toric context.[5] For each ray v_i in the fan, there is a toric Weil divisor D_i. The line bundles can then be expressed as $\mathcal{O}\left(\sum_i a_i D_i\right)$ for $a_i \in \mathbb{Z}$. One very special line bundle is the anti-canonical bundle:

$$\mathcal{O}(-K) = \mathcal{O}\left(\sum_i D_i\right) . \tag{2.1}$$

As we said earlier, the Calabi-Yau cone is the total space of the canonical bundle over our surface. The fact that our fan defines a convex polygon means that K is negative.

To work with exceptional collections, we need to compute $\dim \mathrm{Ext}^q(E_i, E_j)$. We are dealing with surfaces, and so the q's of interest are $q = 0$, 1, or 2. For line bundles, the Ext groups can be understood in terms of sheaf valued cohomology,

$$\mathrm{Ext}^q_X(\mathcal{O}(E), \mathcal{O}(F)) = H^q(X, \mathcal{O}(F - E)) . \tag{2.2}$$

Serre duality[6] is useful in understanding what the non-vanishing Ext groups are

$$H^q(X, \mathcal{O}(E)) = H^{2-q}(X, \mathcal{O}(-E + K))^\vee . \tag{2.3}$$

There is a straightforward way to compute $\dim \mathrm{Hom}(E_i, E_j)$ in the context of toric geometry if E_i and E_j are line bundles [22]. This dimension is the number of global sections of the difference line bundle $E_j \otimes E_i^*$, which we denote by $\mathcal{O}(D) = \mathcal{O}(\sum_i a_i D_i)$. To compute $\dim \mathrm{Hom}(E_i, E_j)$ one constructs the polygon

$$\Delta_D = \{u \in \mathbb{R}^2 : u \cdot v_i \geq -a_i\} . \tag{2.4}$$

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[5]For singular surfaces when D is not a Cartier divisor, $\mathcal{O}(D)$ is actually not a line bundle but only a reflexive sheaf. Nevertheless, for the sake of brevity we don't emphasize this anymore.

[6]Every toric variety is Cohen-Macaulay [22], and therefore the Grothendieck-Serre duality theory simplifies: Grothendieck's dualizing complex is a just the canonical sheaf.

for a two vector $u = (x, y)$. The dimension $\dim \operatorname{Hom}(E_i, E_j)$ is the number of lattice points in this polygon. Although often tedious to compute by hand in all but the simplest cases, it is easy to implement a point counting routine on a computer. Through Serre duality, this point counting routine allows one to compute $\dim \operatorname{Ext}^2(E_i, E_j)$ as well.

We also need to compute $\dim \operatorname{Ext}^1(E_i, E_j)$. There exist certain vanishing theorems which tell us when the higher cohomology vanishes. For example, Kodaira's vanishing theorem says that given a line bundle $\mathcal{O}(D)$ corresponding to an ample divisor D, then[7]

$$\dim H^q(X, \mathcal{O}(D \otimes K)) = 0, \qquad \text{for any } q > 0. \tag{2.5}$$

There is a straightforward way to test whether a toric divisor $D = \sum a_i D_i$ is ample. Construct the piecewise linear function $\psi_D(v)$ where v is a real valued vector in the space spanned by the v_i and the function ψ_D takes the values $\psi_D(v_i) = -a_i$. If ψ_D is strictly convex, then D is ample [22].

In fact there is another useful result connected with this function ψ_D. If ψ_D is convex, then the higher cohomology of $\mathcal{O}(D)$ itself vanishes: $H^q(X, \mathcal{O}(D)) = 0$ for $q > 0$. However, since for us K is negative, Kodaira vanishing is stronger than this result. To keep things simple, in constructing a strong exceptional collection, we would like $E_i^* \otimes E_j$ to be a line bundle corresponding to an ample divisor.

We have implemented this lattice point counting routine and Kodaira vanishing via ψ_D on the computer algebra package Maple. In the simple and especially the smooth cases, this Maple routine is usually enough to tell whether the collection is strongly exceptional. However, in certain of the more complicated singular examples, such as the $L^{p,q,r}$, Kodaira vanishing is not enough to guarantee that $H^1 = 0$. In this case, we need to work harder.

In the appendix, we describe a more elaborate procedure based on local cohomology for calculating all of H^q, including H^1, using Macaulay2. This procedure is based on results of [17].

The exceptional collection remains exceptional (and generates the same quiver, since the quiver only depends on differences) if we tensor every object by the same line bundle. Physically this corresponds to large radius monodromy. For example, $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$, and $\mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3)$ are both exceptional collections on $\mathbb{P}^2$. Therefore, let us choose $E_1 = \mathcal{O}$. Serre duality and Kodaira vanishing suggest a sense in which the collection is sandwiched between $\mathcal{O}$ and $\mathcal{O}(-K)$.

Given a strongly exceptional collection $\mathcal{E}$, the quiver gauge theory can be constructed from the inverse collection $\mathcal{E}^\vee$ (these are the fractional branes). The inverse collection is constructed by mutation, and is inverse in the Euler character sense, $\chi(E_i, E_j^\vee) = \delta_{ij}$. The inverse collection is also exceptional although no longer strongly exceptional. The Euler character $\chi(E_i^\vee, E_j^\vee)$ can be interpreted as the number of arrows from node i to node j in the quiver minus the number of arrows from node j to node i [12, 21].

In the smooth case, a result of Kuleshov and Orlov [23] guarantees that there is at most one map between any two objects in an exceptional collection and moreover this map

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[7]In the cases where X has no worse than log terminal singularities (i.e. most of the cases discussed in this paper), there is a Kawamata-Viehweg vanishing theorem generalizing Kodaira vanishing.

is either Ext^0 or Ext^1. This result appears no longer to be true in the singular case, and we have examples below. As a result, computing χ only gives the quiver up to bidirectional arrows. However, most of the time χ is enough.

Recently, Kawamata has given a construction for generating exceptional collections for any toric variety [24]. However, his exceptional collections are typically not strong and also involve torsion sheaves. Although we borrow his results for weighted projective space, for more complicated examples, we follow our own simpler, but not so general, methods.

Presently, we review how to derive exceptional collections for some smooth toric surfaces, but first, we make a few remarks about ordering and the canonical class.

2.3 Ordering the sheaves

In the smooth (del Pezzo) case, the Kuleshov and Orlov result [23] discussed above means it is enough to compute the Euler character to determine the maps between the objects of an exceptional collection,

$$\chi(E, F) = \sum_k (-1)^k \dim \text{Ext}^k(E, F) \ . \tag{2.6}$$

For a smooth surface $\mathcal{B}$, Riemann-Roch implies that

$$\chi(E, F) = \int_{\mathcal{B}} \text{ch}(E^*) \, \text{ch}(F) \, \text{Td}(\mathcal{B}) \tag{2.7}$$

$$= \text{rk}(E) \, \text{rk}(F) + \frac{1}{2} (\text{rk}(E) \deg(F) - \text{rk}(F) \deg(E)) +$$
$$\text{rk}(E) \, \text{ch}_2(F) + \text{rk}(F) \, \text{ch}_2(E) - c_1(E) \cdot c_1(F)$$

where we have used the Chern character of the sheaf

$$\text{ch}(E) = (\text{rk}(E), c_1(E), \text{ch}_2(E)) \ . \tag{2.8}$$

Also, $\text{Td}(\mathcal{B}) = 1 - \frac{K}{2} + H^2$, where K is the canonical class and H is the hyperplane, with $\int_{\mathcal{B}} H^2 = 1$. Finally the degree $\deg(E) = (-K) \cdot c_1(E)$.

We denote $\chi_-(E, F)$ to be the antisymmetric part of χ

$$\chi_-(E, F) = \text{rk}(E) \deg(F) - \text{rk}(F) \deg(E) \ . \tag{2.9}$$

Note that for an exceptional pair (E, F), $\chi_-(E, F) = \chi(E, F)$.

From this last result, we can conclude that for a strongly exceptional collection, if $i > j$, then

$$\frac{\deg(E_i)}{\text{rk}(E_i)} \geq \frac{\deg(E_j)}{\text{rk}(E_j)} \ . \tag{2.10}$$

Otherwise, we would get Ext^1 maps. This inequality motivates the definition of slope, $\mu(E_i) = \deg(E_i)/\text{rk}(E_i)$. This way the sheaves in a strongly exceptional collection are ordered by their slope. If we further insist that all the sheaves are line bundles $(\text{rk}(E) = 1)$, as we will do for most of this paper, the sheaves are ordered by their degree.

Although Riemann-Roch is substantially more complicated for our singular examples, we have found on an example by example basis, given a strongly exceptional collection of line bundles, the bundles are strongly exceptional ordered by their degree.

We would like to introduce the notion of a helix before moving on to the examples in order to try to bring a different point of view to the sandwiching of the collection mentioned above. In the smooth case, Bondal and others [5] have shown that given an exceptional collection

$$\mathcal{E} = (E_1, E_2, \ldots, E_n) \, , \tag{2.11}$$

we can generate another exceptional collection by right mutating E_1 all the way to the end, $E_1 \to R_{E_n} \cdots R_{E_3} R_{E_2} E_1$. Moreover, this right mutated sheaf is

$$R_{E_n} \cdots R_{E_3} R_{E_2} E_1 = E_1 \otimes (-K) \, . \tag{2.12}$$

This process can be inverted and repeated, generating a bi-infinite sequence of sheaves called a helix, any n adjacent elements of which form an exceptional collection. We call any such n adjacent elements a foundation for the helix. In [20], it was shown that given a helix, any choice of foundation generates the same quiver.

Above we asserted that we needed a strong exceptional collection to generate a physical gauge theory quiver. In light of the existence of a helix, the requirement is actually a little stronger. The strong exceptional collection must generate a helix such that any foundation is also strongly exceptional [25]. Such a helix is said to be strong.

From our Euler character formula, it is straightforward to see that a strong exceptional collection in the smooth case generates a strong helix if and only if

$$\mu(E_n) \geq \mu(E_1) + K^2 \, . \tag{2.13}$$

In the singular case, we have examples where this inequality does not hold but we still get physical quivers. Nonetheless, the inequality remains a good guideline, in the sense that if the inequality is not satisfied and the collection is not strongly exceptional, then the collection is improved if some sheaves can be moved inside this window. Given a strongly exceptional collection which does not satisfy this inequality, in the singular cases we have studied, there is always a related strongly exceptional collection which does and which produces the same quiver.

3. Smooth toric surfaces

Exceptional collections for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up at one, two or three points can all be constructed by following a remarkably simple rule. Given a fan $\Delta = \{v_i \in \mathbb{Z}^2 : i = 1, \ldots, n\}$ such that the v_i are presented in clockwise or counterclockwise order, there exists an exceptional collection of the form[8]

$$\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(D_1 + D_2), \ldots, \mathcal{O}\left(\sum_{i=1}^{n-1} D_i\right). \tag{3.1}$$

Let's see how this rule works for each of the smooth Fano toric surfaces listed above.

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[8]This collection was described in [26].

3.1 $\mathbb{P}^2$

We can write the fan in the form $v_1 = (1,0)$, $v_2 = (0,1)$, and $v_3 = (-1,-1)$. These three vectors satisfy the relation $v_1 + v_2 + v_3 = 0$, and all three divisors are linearly equivalent. Thus, the collection $\mathcal{O}$, $\mathcal{O}(D_1)$, $\mathcal{O}(D_1 + D_2)$ can be written in the more familiar form $\mathcal{O}$, $\mathcal{O}(H)$, $\mathcal{O}(2H)$ given above.

For this collection the $\mathrm{Ext}^q(E_i, E_j)$ groups vanish for any $q > 0$.. The matrix $S_{ij} = \dim \mathrm{Hom}(E_i, E_j)$ takes the form

$$S = \begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{3.2}$$

From the inverse of this matrix, we can construct the gauge theory quiver

$$S^{-1} = \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{3.3}$$

The above-diagonal entries of S_{ji}^{-1} can be reinterpreted as the number of arrows from node i to node j in the quiver. A negative sign means the orientation of the arrow should be reversed.

We will discuss $\mathbb{P}^2$ blown up at one point and $\mathbb{P}^1 \times \mathbb{P}^1$ later, and in a slightly different way, which will be useful for our singular examples. So, to avoid repetition, we jump directly to dP_2.

3.2 $\mathbb{P}^2$ blown up at two points: dP_2

A fan for dP_2 is given by

$$v_1 = (1,0), \; v_2 = (0,1), \; v_3 = (-1,0), \; v_4 = (-1,-1), \; v_5 = (0,-1) \,. \tag{3.4}$$

Using our rule, there is an exceptional collection of the form

$$(0,0,0,0,0), (1,0,0,0,0), (1,1,0,0,0), (1,1,1,0,0), (1,1,1,1,0) \,, \tag{3.5}$$

where we have introduced the shorthand notation

$$\mathcal{O}(\sum_i a_i D_i) \equiv (a_1, a_2, \ldots, a_n). \tag{3.6}$$

The higher Ext groups vanish for this example and we find that

$$S = \begin{pmatrix} 1 & 2 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.7}$$

from which we calculate

$$S^{-1} = \begin{pmatrix} 1 & -2 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.8}$$

The quiver corresponding to S^{-1} is precisely case II from page 13 of [27].

3.3 dP_3

The fan in this case is

$$(1, 0), (1, 1), (0, 1), (-1, 0), (-1, -1), (0, -1) \tag{3.9}$$

from which we construct the collection

$$(0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0), (1, 1, 0, 0, 0, 0), \dots, (1, 1, 1, 1, 1, 0) . \tag{3.10}$$

From this collection, we find that the higher Ext groups vanish while

$$S = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.11}$$

The inverse matrix is then

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.12}$$

The corresponding quiver was first written down using toric methods, and is Model I of Beasley and Plesser [16].

4. Weighted projective spaces

Let's consider the weighted projective space $\mathbb{P}(1, a, b)$ defined by the complete fan $v_1 = (-a, -b)$, $v_2 = (1, 0)$ and $v_3 = (0, 1)$. These three vectors satisfy the relation $v_1 + av_2 + bv_3 = 0$, and thus the toric surface can be thought of as a $\mathbb{C}^*$ quotient of $\mathbb{C}^3 - \{0\}$, where the $\mathbb{C}^*$ acts with weights $(1, a, b)$ on the three coordinates.

The total space of the canonical bundle $\mathcal{O}(K)$ over $\mathbb{P}(1, a, b)$ is a $\mathbb{Z}_{1+a+b}$ orbifold of $\mathbb{C}^3$ with weights $(1, a, b)$. String theorists (see for example [28]) associate a quiver to such a

$\mathbb{Z}_{a+b+1}$ orbifold singularity. The quiver consists of $a+b+1$ nodes and a collection of arrows between the nodes. In particular, node i will have three outgoing arrows, one X_i ending on node $i+1$, one Y_i ending on node $i+a$ and one Z_i ending on node $i+b$. The quiver also has a number of relations. In particular, consider two different paths that begin and end on the same nodes. These paths are equivalent if the two paths have the same number of X's, the same number of Y's, and the same number of Z's.

We intend to rederive this quiver using an exceptional collection. Kawamata [24] shows that a strong exceptional collection on $\mathbb{P}(1, a, b)$ will consist of all possible objects of the form

$$\mathcal{O}\left(k_1 D_1 + k_2 D_2 + k_3 D_3\right) , \tag{4.1}$$

such that $k_i \geq 0$ and $0 \leq k_1 + ak_2 + bk_3 < 1 + a + b$. Moreover, given two objects E and E' in the collection such that E comes before E', we must have $k_1 + ak_2 + bk_3 < k_1' + ak_2' + bk_3'$. Thus, an exceptional collection on $\mathbb{P}(1, a, b)$ can always be written as

$$\mathcal{O}, \mathcal{O}(D_1), \mathcal{O}(2D_1), \ldots, \mathcal{O}((a + b)D_1) . \tag{4.2}$$

Note that the divisors satisfy the linear equivalence relations $aD_1 \sim D_2$ and $bD_1 \sim D_3$. The same strongly exceptional collection is obtained by the authors of [29].

Let's see how this claim works in some specific examples.

4.1 $\mathbb{P}(1,1,2)$

This $\mathbb{Z}_4$ orbifold has an exceptional collection

$$(0,0,0), (1,0,0), (1,1,0), (2,1,0) \tag{4.3}$$

where again we use the shorthand notation $\mathcal{O}(k_1 D_1 + k_2 D_2 + k_3 D_3) \equiv (k_1, k_2, k_3)$. From this collection the matrix $S_{ij} = \dim \mathrm{Hom}(E_i, E_j)$ is

$$S = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.4}$$

From the gauge theory quiver for this $\mathbb{Z}_4$ orbifold, we can construct the so-called Beilinson quiver on the toric surface $\mathbb{P}(1,1,2)$ by cutting all the arrows that point from node i to node j where $i > j$ (see figure 1). These cut arrows are then reinterpreted as relations among the remaining maps. The dimensions in S are the number of ways of getting from node i to node j taking into account the relations.

Once again, the nonzero above the diagonal entries S_{ij}^{-1} (for $i < j$) are reinterpreted as the number of arrows from j to i minus the number of arrows from i to j. For $\mathbb{P}(1,1,2)$, we find

$$S^{-1} = \begin{pmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$
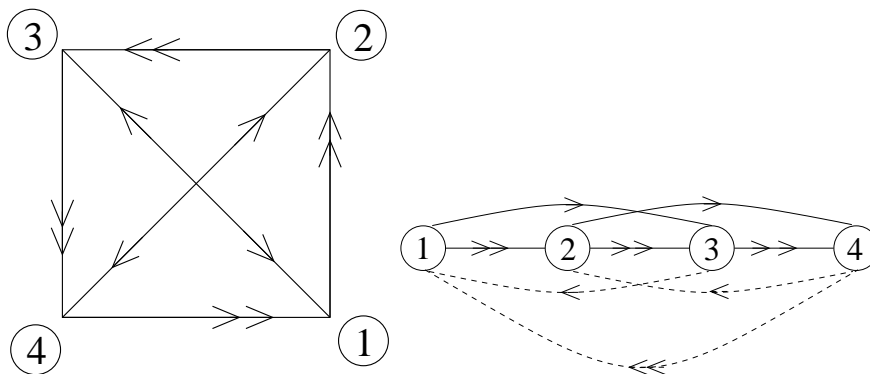
which gives the quiver in figure 1.

Figure 1: The gauge theory quiver and the Beilinson quiver for $\mathbb{P}(1,1,2)$.

Of course, S^{-1} doesn't show that there are in fact arrows running in opposite directions between nodes 1 and 3 and also between nodes 2 and 4. In this computation S_{ij} is the Euler character $\chi(E_i, E_j)$ because the collection is strong. S_{ji}^{-1} is then the Euler character $\chi(E_i^\vee, E_j^\vee)$ of the dual gauge theory collection [12, 20, 25], which is not strong.

4.2 $\mathbb{P}(1,2,3)$

This $\mathbb{Z}_6$ orbifold has an exceptional collection

$$(0,0,0), (1,0,0), (2,0,0),$$
$$(1,1,0), (1,0,1), (2,0,1). \tag{4.6}$$

The S_{ij} matrix here is

$$S = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.7}$$
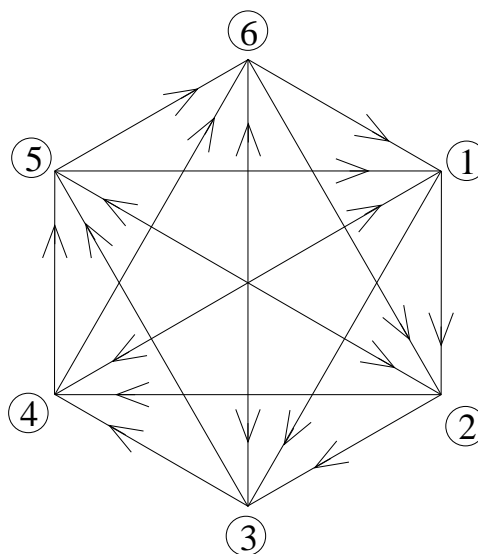

S^{-1} then leads to the quiver in figure 2. Going the other way, it is again straightforward to reconstruct S_{ij} from the Beilinson quiver.

Figure 2: The $\mathbb{P}(1,2,3)$ gauge theory quiver.

4.3 $\mathbb{P}(1,a,a)$

The weighted projective space $\mathbb{P}(1,a,a)$ can be thought of in two different ways. As a scheme or as a stack.[9] Since the singularities at $(0 : x_2 : x_3)$ form a codimension one subspace, this space is not singular, even as a scheme, and hence $\mathbb{P}(1,a,a) \cong \mathbb{P}^2$. The origin of this "smoothing-out" is identical to the one that underlies the isomorphism $\mathbb{C}/\mathbb{Z}_n \cong \mathbb{C}$.

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[9]For a recent discussion see [29].

Algebraically this isomorphism is in fact a trivial statement: $\operatorname{Spec}\mathbb{C}[x^n] \cong \operatorname{Spec}\mathbb{C}[x]$. We will return to this issue at length in section 6.

The traditional language of toric geometry [22] is not well suited to distinguish between $\mathbb{P}(1,a,a)$ and $\mathbb{P}^2$, and there are two different formalisms which can be introduced to make this distinction. The first is that of boundary divisors. We add the extrinsic information that there is a codimension one singularity along the divisor D_1, which branches a times: $\mathbb{P}(1,a,a)$ is equivalent to $\mathbb{P}^2$ with boundary divisor $(a-1)D_1/a$ [24]. The second, more powerful formalism, is the recently discovered theory of *toric Deligne-Mumford stacks* of Borisov, Chen and Smith [19]. Regardless of the formalism, from the string theory point of view, this singular surface is embedded in a Calabi-Yau space, the total space of its canonical sheaf, and the singularity structure is unambiguous. It is only when we try to analyze the $3d$ problem with $2d$ tools that we have to use more powerful techniques.

Although for the proofs in section 6 we used stacky technology, for our Macaulay2 and Maple routines the fact that two of the weights have a gcd larger than one posed no extra complication. The algorithms are the same when computing the dimensions of the cohomology of varieties with or without codimension one singularities, although the interpretations differ.

The collection on $\mathbb{P}(1,a,a)$, which will be useful in the next section, is

$$(0,0,0),(1,0,0),(2,0,0),\ldots,(a-1,0,0),$$
$$(0,0,1),(1,0,1),(2,0,1),\ldots,(a-1,0,1) .$$
(4.8)

5. Beyond weighted projective spaces

As we alluded to in the introduction, at the moment we have only two tools in our basket for deriving exceptional collections for toric surfaces that are generated by more than three rays. The first is a divisorial blow-up and the second is an exterior tensor product.

For the divisorial blow-up, we start with two toric surfaces X and Y related by the blow-up map $\pi\colon X \to Y$. Assuming that

$$\mathcal{E} = \mathcal{O}_Y, E_2, E_3, \ldots E_n$$

is an exceptional collection on Y, we would like to figure out the collection on X.

Let the fan for Y have m rays. The blow-up corresponds to adding a primitive ray v_{m+1} to the fan, and also gives a new toric divisor D_{m+1}. The new exceptional collection is taken to be

$$\mathcal{E}' = \mathcal{O}_X, \mathcal{O}_X(D_{m+1}), \pi^*E_2, \ldots, \pi^*E_n.$$

But we can be much more explicit. We organize the rays in a counterclockwise fashion, and let's assume that the blow-up corresponds to the relation $a_m v_m + a_1 v_1 = a_{m+1} v_{m+1}$ for some integers a_i. From a theorem of Kawamata [24],[10] if $E_j = (k_1, k_2, \ldots, k_m)$, then

$$\pi^*E_j = \left(k_1, k_2, \ldots, k_m, \lfloor \frac{a_m k_m + a_1 k_1}{a_{m+1}} \rfloor\right) .$$
(5.1)

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[10]We will implicitly prove this theorem in section 6.3.

For a smooth blow-up $a_m = a_1 = a_{m+1} = 1$, and we will show that the procedure generates a strongly exceptional collection.

For the tensor product case we assume that the fan decomposes into $\mathbb{P}^1$ and something for which we already know the exceptional collection. For example, the fan consists of $v_1, v_2, \ldots v_m, (1,0), (-1,0)$ and we already know an exceptional collection $\mathcal{E}$ for the $v_1, v_2, \ldots v_m$ part. Let

$$\mathcal{E} = (k_1^1, k_2^1, \ldots, k_m^1), (k_1^2, k_2^2, \ldots, k_m^2), \ldots, (k_1^n, k_2^n, \ldots, k_m^n), \tag{5.2}$$

for some collection of integers k_j^i. Then the collection on the whole space is

$$\begin{aligned} \mathcal{E}' = &(k_1^1, k_2^1, \ldots, k_m^1, 0, 0), (k_1^2, k_2^2, \ldots, k_n^2, 0, 0), \ldots, (k_1^n, k_2^n, \ldots, k_m^n, 0, 0), \\ &(k_1^1, k_2^1, \ldots, k_m^1, 1, 0), (k_1^2, k_2^2, \ldots, k_m^2, 1, 0), \ldots, (k_1^n, k_2^n, \ldots, k_m^n, 1, 0). \end{aligned}$$

5.1 Two easy examples: $\mathbb{F}_0$ and dP_1

Here are two examples of smooth toric surfaces to which we can apply the techniques described above, in a context where we already know the answers.

Take $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ for which the fan is $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,0)$, and $v_4 = (0,-1)$. An exceptional collection on a single $\mathbb{P}^1$, described by v_1 and v_3, is just $(0,0,0,0), (1,0,0,0)$. Using our procedure described above, a collection on the whole space is then

$$(0,0,0,0), (1,0,0,0), (0,1,0,0), (1,1,0,0). \tag{5.3}$$

Often this collection is written in a slightly different way. Since $D_1 \sim D_3$ and $D_2 \sim D_4$, we can suppress the last two entries and write instead $\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)$, the well known three-block exceptional collection on $\mathbb{P}^1 \times \mathbb{P}^1$.

Take $\mathbb{P}^2$ blown up at a point, also known as the first del Pezzo surface, dP_1, or the first Hirzebruch surface $\mathbb{F}_1$. The fan for $\mathbb{P}^2$ is $v_1 = (0,-1)$, $v_2 = (1,1)$, $v_3 = (-1,0)$ to which we add the primitive ray $v_4 = (-1,-1)$. A well known collection on $\mathbb{P}^2$ is usually written $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ which we translate into our toric notation as

$$(0,0,0), (0,0,1), (0,1,1). \tag{5.4}$$

Running our procedure, the collection on dP_1 should be

$$(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,1,1,1). \tag{5.5}$$

Let H be the hyperplane divisor of $\mathbb{P}^2$ and E the exceptional divisor on dP_1. With a little work, one can show that $D_1 \sim D_3 \sim H - E$, $D_4 \sim E$, and $D_2 \sim H$. Then this collection can be written in the more familiar notation

$$\mathcal{O}, \mathcal{O}(E), \mathcal{O}(H), \mathcal{O}(2H). \tag{5.6}$$

5.2 Singular examples

In this and the following subsections, we will be interested in two large classes of examples. The first are the $Y^{p,q}$ spaces [30], $0 \leq q \leq p$, which have the cone generated by the vectors

$$V_1 = (1,0,0), \quad V_2 = (1,1,0), \quad V_3 = (1,p,p), \quad V_4 = (1,p-q-1,p-q) \qquad (5.7)$$

and the $X^{p,q+1}$ [18] which have the additional vector

$$V_5 = (1,p-q-2,p-q-1) . \qquad (5.8)$$

We start out by considering the simplest $Y^{p,q}$ space, namely $q = p$.

5.2.1 $Y^{p,p}$

In this case, we can drop the $V_1 = (1,0,0)$ vector because it lies on the line joining V_2 and V_4. We then convert this cone into a two dimensional fan by choosing one of the interior points to be the origin. We depicted a typical example in figure 3, indicating all the interior lattice points.

If we choose $(1, p-1, p-1)$ as the origin, the 2d fan is given by $v_2 = (-p+2, -p+1)$, $v_3 = (1,1)$, and $v_4 = (-p, -p+1)$, which satisfies the single relation $v_2 + v_4 + (2p-2)v_3 = 0$. In other words, we recover the weighted projective space $\mathbb{P}(1, 2p-2, 1)$ that we expect from [30, 31] and have already treated in section 4. The cone over $Y^{p,p}$ is well known to be the orbifold $\frac{1}{2p}(1, 2p-2, 1) = \frac{1}{2p}(1, -2, 1)$.

But we could as well choose any of the interior points $(1, i, i)$, for any $1 \leq i \leq p-1$, as the origin of the 2d fan (see figure 3). For example, choosing $(1,1,1)$ we arrive at the projective space $\mathbb{P}(p-1, p-1, 2)$.

Whether we choose the origin $(1, p-1, p-1)$ or $(1,1,1)$, the gauge theories had better be the same. For p even, this is easy to see. With the origin $(1, p-1, p-1)$ we had the $\frac{1}{2p}(1, 1, -2)$ orbifold of $\mathbb{C}^3$. In the second case, with $(1,1,1)$ as the origin, we got the $\frac{1}{2p}(p-1, p-1, 2)$ orbifold. But these two actions are the same: by our assumption p was
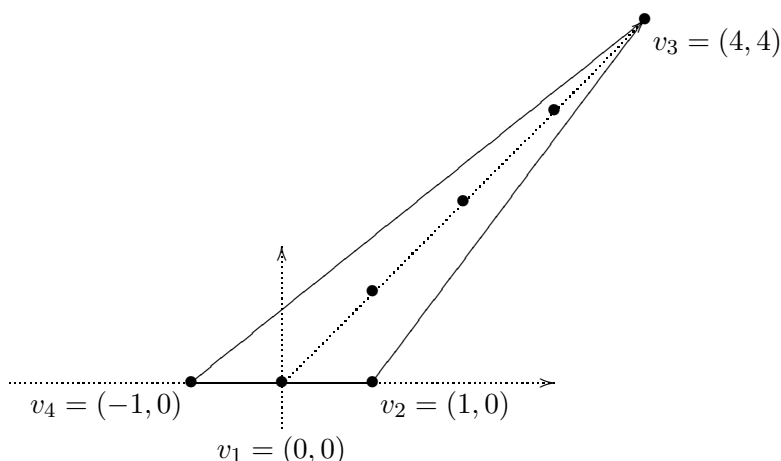

Figure 3: The toric fan of $Y^{4,4}$.

even and hence $\gcd(2p, p-1) = 1$, and thus the two spaces differ only by a choice of the $\mathbb{Z}_{2p}$ generator (if the first generator is ω, the generator of the second orbifold is ω^{p-1}).

When p is odd, the situation is more subtle. With choice of origin $(1,1,1)$, the fan is $v_2 = (0,-1)$, $v_3 = (p-1, p-1)$, and $v_4 = (-2,-1)$. These v_i do not span our $\mathbb{Z}^2$ lattice, covering only every other point. Thus, the space is not $\mathbb{P}((p-1)/2, (p-1)/2, 1)$ but a $\mathbb{Z}_2$ orbifold of it. Fortunately, Kawamata [24] tells us how to handle this case as well. We begin by writing the usual exceptional collection for $\mathbb{P}((p-1)/2, (p-1)/2, 1)$:

$$\mathcal{O}, \mathcal{O}(D_3), \mathcal{O}(2D_3), \ldots, \mathcal{O}((p-1)D_3) \ . \tag{5.9}$$

On this $\mathbb{Z}_2$ orbifold, there is also a torsion divisor, $(p-1)D_3/2 - D_4$, of order 2: $(p-1)D_3 - 2D_4$ is linearly equivalent to zero. Under the orbifolding, a line bundle $\mathcal{O}(D)$ on $\mathbb{P}((p-1)/2, (p-1)/2, 1)$ pushes forward to a direct sum $\mathcal{O}(D) \oplus \mathcal{O}(D + (p-1)D_3/2 - D_4)$, and Kawamata proves (Theorem 3.5 of [24]) that the collection

$$\mathcal{O}, \mathcal{O}((p-1)D_3/2 - D_4), \mathcal{O}(D_3), \mathcal{O}((p+1)D_3/2 - D_4), \ldots,$$

$$\mathcal{O}((p-1)D_3), \mathcal{O}(3(p-1)D_3/2 - D_4) \tag{5.10}$$

is strongly exceptional on the orbifold.

Let's consider the case $p = 3$. The choice of origin $(1,2,2)$ leads to the weighted projective space $\mathbb{P}(1,1,4)$ and the expected quiver on this $\mathbb{C}^3/\mathbb{Z}_6$ orbifold. However, if we choose $(1,1,1)$ for origin, then we need to use the exceptional collection (5.10). From our Maple routine, it is easy to compute

$$S^{-1} = \begin{pmatrix} 1 & 0 & -1 & -2 & 1 & 2 \\ 0 & 1 & -2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.11}$$

which, up to reordering of the nodes, yields the same quiver as the $\mathbb{Z}_6$ orbifold with weights $(1,1,4)$.

The choice of origin has a nice geometric interpretation. Adding the extra interior point blows up a divisor in the Calabi-Yau, partially desingularizing it. Going from one interior point to another is a birational transformation composed of a blow-down followed by a blow-up. Since this is a Kähler deformation, we have reasons to expect that the derived categories are equivalent. This has already been proven for the conifold [32], and more generally for quotient stacks [33], but our case is even more general.

In practice, we try to avoid blowing up points which lead to this extra subtlety of torsion divisors. The reason is that our Macaulay2 program cannot handle torsion divisors directly. More generally, this subtlety occurs whenever the fan only spans a sublattice of the $\mathbb{Z}^2$ lattice on which it is written.

5.2.2 $Y^{p,p-1}$

Next we treat $q = p - 1$. Choosing the origin $(1, 1, 1)$, we find the fan

$$v_1 = (-1, -1), \ \ v_2 = (0, -1), \ \ v_3 = (p - 1, p - 1), \ \ v_4 = (-1, 0) \,. \qquad (5.12)$$

In this case, v_2, v_3 and v_4 form the weighted projective space $\mathbb{P}(p - 1, 1, p - 1)$, while v_1 is the exceptional divisor of a blow-up. Note that the point we blew up, corresponding to the large cone $\{v_2, v_4\}$, was smooth. From our construction, we expect an exceptional collection for $Y^{p,p-1}$ to be

$$\begin{aligned} \mathcal{E} = \ & (0,0,0,0), (1,0,0,0), (0,0,1,0), (0,0,2,0), \ldots, (0,0,p-2,0), \qquad (5.13) \\ & (1,0,0,1), (1,0,1,1), (1,0,2,1), \ldots, (1,0,p-2,1), (2,1,0,1) \ . \end{aligned}$$

To familiarize ourselves with the $Y^{p,p-1}$ case we consider the first four: $Y^{1,0}$, $Y^{2,1}$, $Y^{3,2}$ and $Y^{4,3}$. The simplest non-trivial $Y^{p,p-1}$ space is $Y^{1,0}$. By (5.12) the Calabi-Yau variety $Y^{1,0}$ is "the" conifold, first studied in the AdS/CFT context in [1, 34, 35]. The $Y^{1,0}$ case is degenerate from the $Y^{p,p-1}$ point of view, since it has no interior lattice points, and its desingularization is a small resolution.

The next in line, $Y^{2,1}$, is the cone over dP_1, which we already considered in section 5.1. So let us consider the next two examples of $Y^{p,p-1}$-type.

For $Y^{3,2}$, we find the collection

$$\begin{aligned} & (0,0,0,0), (1,0,0,0), (0,0,1,0), \\ & (1,0,0,1), (1,0,1,1), (2,1,0,1), \end{aligned} \qquad (5.14)$$
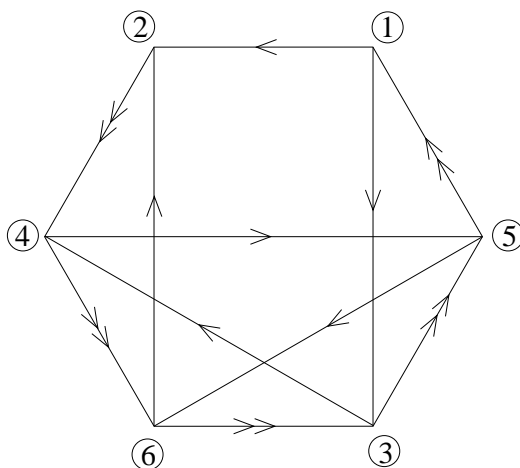

Figure 4: The usual quiver for $Y^{3,2}$.

from which we may compute

$$S = \begin{pmatrix} 1 & 1 & 1 & 3 & 3 & 6 \\ 0 & 1 & 0 & 2 & 2 & 5 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.15}$$

and

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5.16}$$

Using S_{ij}^{-1}, we may compute the usual quiver for $Y^{3,2}$ (see figure 4). Note this quiver is well split [25] given the cyclic ordering from the exceptional collection. Well split means that if we order the nodes cyclically such that node $n+1$ is equivalent to node 1, all the incoming arrows to node i come from nodes $i-1$, $i-2$, etc. and all the outgoing arrows go to nodes $i+1$, $i+2$, etc. Seiberg duality is so far understood as a mutation only for well split nodes.

Now onto $Y^{4,3}$: the collection (5.13) gives the following matrices for $Y^{4,3}$

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 6 \\ 0 & 1 & 0 & 0 & 2 & 2 & 2 & 5 \\ 0 & 0 & 1 & 1 & 1 & 3 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.17}$$

and

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.18}$$

which yields the single impurity quiver for $Y^{4,3}$ (figure 4 of [30]).

5.2.3 $Y^{p,p-2}$

Next we consider the example $Y^{p,p-2}$. Choosing again the origin of the fan $(1,1,1)$, we get

$$v_1 = (-1,-1), \quad v_2 = (0,-1),$$
$$v_3 = (p-1,p-1), \quad v_4 = (0,1). \tag{5.19}$$

For these surfaces, the total space is a direct product: $\mathbb{P}^1 \times \mathbb{P}^1(1, p-1)$. The rays v_2 and v_4 "form" a $\mathbb{P}^1$, while v_1 and v_3 give the weighted projective space. We can write an exceptional collection for the orbifolded $\mathbb{P}^1$ using the weighted projective space techniques described above. The exceptional collection on the whole space is then

$$\mathcal{E} = (0,0,0,0), (0,0,1,0), (0,0,2,0), \ldots, (0,0,p-2,0), (1,0,0,0), \tag{5.20}$$
$$(0,0,0,1), (0,0,1,1), (0,0,2,1), \ldots, (0,0,p-2,1), (1,0,0,1) .$$

The simplest example is $Y^{2,0}$, which in fact is the cone over $\mathbb{F}_0$, as discussed in section 5.1. The next simplest example is $Y^{3,1}$, for which we find the exceptional collection

$$(0,0,0,0), (0,0,1,0), (1,0,0,0), (0,0,0,1), (0,0,1,1), (1,0,0,1) . \tag{5.21}$$

From the collection, we compute

$$S = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 4 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.22}$$

and

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & -2 & 2 & 2 \\ 0 & 1 & -1 & 0 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5.23}$$

This S^{-1} gives rise to the double impurity quiver for $Y^{3,1}$ [36] (see figure 5). Seiberg dualizing on node three or four yields the quiver with two single impurities. Seiberg duality corresponds to mutating E_3 all the way to the right or E_4 all the way to the left.

The next simplest case is $Y^{4,2}$. The collection presented in (5.20) gives the following matrix

$$S^{-1} = \begin{pmatrix} 1 & -1 & 0 & -1 & -2 & 2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.24}$$
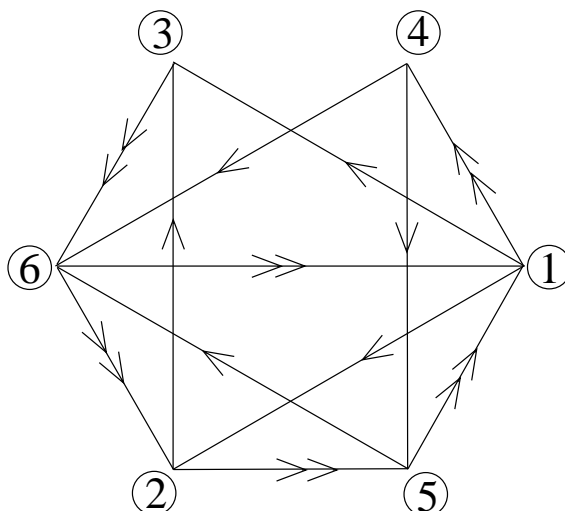
Figure 5: The double impurity quiver for $Y^{3,1}$.

which yields the standard double impurity quiver for $Y^{4,2}$ (figure 2 of [36]).

We can also perform a Seiberg duality on this quiver to get two single impurities, left mutating E_5 all the way to the left of the collection, yielding

$$E_5' = L_{E_1}E_5 = (0,0,0,-1).$$
(5.25)

Tensoring the whole collection by (0,0,0,1) then yields

$$(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,0,2,1), (1,0,0,1), (0,0,1,2), (0,0,2,2), (1,0,0,2) .$$
(5.26)

From this collection we find

$$S^{-1} = \begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
(5.27)

which gives a quiver with two single impurities and is well split (figure 6 of [30] or figure 2 of [36]).

5.3 Some conjectures

While we were able to motivate the collections for $Y^{p,p-1}$ and $Y^{p,p-2}$, here we are doing something closer to guesswork, extrapolating results for small p.

For $Y^{p,p-3}$, and $Y^{p,p-4}$, the collection

$$(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,0,2,1), \ldots (0,0,(p-2),1),$$
$$(0,0,1,2), (0,0,2,2), \ldots (0,0,(p-1),2), (0,0,(p-1),3) \tag{5.28}$$

appears to generate appropriate quivers for all p. In particular, for $Y^{p,p-4}$, the collection generates a quiver with two single impurities separated by a double impurity. For $Y^{p,p-3}$, the quiver has three adjacent single impurities. For $Y^{p,p-2}$, the quiver has two single impurities separated by a normal unit.

We can look at an explicit example, say $Y^{4,1}$. The fan is given by

$$v_1 = (-1,-1), v_2 = (0,-1), v_3 = (3,3), v_4 = (1,2) \tag{5.29}$$

and the collection (5.28) reads

$$(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,0,2,1), (0,0,1,2), (0,0,2,2), (1,0,0,1), (1,0,0,2) . \tag{5.30}$$

From this collection, we find

$$S^{-1} = \begin{pmatrix} 1 & -2 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.31}$$

which gives the standard quiver with three single impurities.

For $Y^{p,0}$ and $Y^{p,1}$, the collection

$$(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,0,1,2), \ldots (0,0,p-1,p-1), (0,0,p-1,p) \tag{5.32}$$

appears to be strongly exceptional and generates the appropriate quivers.

Later, we will give a conjectured exceptional collection for any toric surface generated by four rays, including the $Y^{p,q}$ as special cases. With our rules thus far, the simplest $Y^{p,q}$ for which we have not given a collection is $Y^{7,2}$, which has 14 gauge groups!

We also note that all of the $Y^{p,q}$ quivers appear to be well split, a property that does not hold for the $L^{p,q,r}$ spaces we analyze below.

5.4 $X^{p,q}$

Given a collection for $Y^{p,q-1}$, it is a simple matter to generate a collection for $X^{p,q}$. Let $X_{p,q}$ be the surface underlying the 3-fold $X^{p,q}$, i.e., $X^{p,q}$ is the total space of the canonical sheaf on $X_{p,q}$, and similarly let $Y_{p,q-1}$ underlie the 3-fold $Y^{p,q-1}$. We can in fact arrange things in such a way that $X_{p,q}$ is a blow-up of $Y_{p,q-1}$. In particular, choosing (1,1,1) as the

origin of the fan, one finds that $X_{p,q}$ is a blow-up of $Y_{p,q-1}$ at a smooth point. As described at the beginning of section 5, we know how to deal with this case.

The $X_{p,q}$ fan is generated by the vectors

$$
\begin{aligned}
v_1 &= (-1,-1)\,, \quad v_2 = (0,-1)\,, \quad v_3 = (p-1,p-1) \\
v_4 &= (p-q-1,p-q)\,, \quad v_5 = (p-q-2,p-q-1)\,.
\end{aligned}
\tag{5.33}
$$

$Y_{p,q-1}$ is described by the first four vectors, and $X_{p,q}$ is obtained by adding v_5. The smooth blow-up is reflected in the relation

$$
v_5 = v_4 + v_1\,.
\tag{5.34}
$$

Given a strongly exceptional collection

$$
\mathcal{E} = \mathcal{O}_{Y_{p,q-1}}, E_2, E_3, \ldots E_n
$$

on $Y_{p,q-1}$ we will prove in section 6 that collection

$$
\mathcal{E}' = \mathcal{O}_{X_{p,q}}, \mathcal{O}_{X_{p,q}}(D_5), \pi^* E_2, \pi^* E_3, \ldots, \pi^* E_n
$$

is also strongly exceptional.

Let's see how all this works out for the example of X^{42}. The fan in this case is

$$
(-1,-1),(0,-1),(3,3),(1,2),(0,1)\,.
\tag{5.35}
$$

Starting with exceptional collection for $Y^{4,1}$ from (5.30), we find the collection

$$
(00000),(00001),(00011),(00111),(00211),(00122),(00222),(10012),(10023)
\tag{5.36}
$$

for $X^{4,2}$, which is strongly exceptional and yields the known quiver [18].

5.5 Other examples: $L^{p,q,r}$

We now turn to some more complicated examples, the $L^{p,q,r}$ spaces of [37–39].

5.5.1 $L^{1,5,2}$

Consider the cone

$$
V_1 = (1,1,3)\,, \quad V_2 = (1,2,1)\,, \quad V_3 = (1,0,0)\,, \quad V_4 = (1,0,1)\,.
\tag{5.37}
$$

for which we choose the origin (1,1,2). The 2d fan is then generated by

$$
v_1 = (0,1)\,, \quad v_2 = (1,-1)\,, \quad v_3 = (-1,-2)\,, \quad v_4 = (-1,-1)\,.
\tag{5.38}
$$

This fan satisfies the two relations

$$
v_1 + v_3 - v_4 = 0\,, \quad v_2 + v_3 + 3v_1 = 0
\tag{5.39}
$$

We can thus think of the surface as $\mathbb{P}(1,1,3)$ with one smooth point blown up, corresponding to the exceptional divisor D_4. The collection we came up with is

$$(0,0,0,0), (0,0,0,1), (0,0,1,1), (0,0,2,2), (1,0,0,1), (1,0,1,2) \qquad (5.40)$$

which yields

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5.41)$$

in agreement with figure 7 of [37].

5.5.2 $L^{1,7,3}$

The cone for $L^{1,7,3}$ is

$$V_1 = (1,1,0), \quad V_2 = (1,0,1), \quad V_3 = (1,0,2), \quad V_4 = (1,3,3). \qquad (5.42)$$

From this cone, we choose the origin (1,1,1) and the 2d fan

$$v_1 = (0,-1), \quad v_2 = (-1,0), \quad v_3 = (-1,1), \quad v_4 = (2,2). \qquad (5.43)$$

This fan satisfies the two relations $v_1 + v_3 - v_2 = 0$ and $2v_3 + v_4 + 4v_1 = 0$ and thus we think of the surface as $\mathbb{P}(4,2,1)$ blown up at one smooth point. Using our procedure, we write down the collection

$$(0,0,0,0), (0,1,0,0), (0,0,0,1), (0,0,0,2), (0,0,0,3), (0,0,0,4), (0,0,0,5), (0,0,0,6). \qquad (5.44)$$

from which we find

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5.45)$$

in agreement with figure 20 of [37]. We will consider more general $L^{p,q,r}$ spaces in section 7.

6. Proofs

In this section we provide proofs of several of our statements about strongly exceptional collections, for infinitely many families at once. Since the local Calabi-Yau spaces in question are total spaces of canonical sheaves on algebraic surfaces, one wonders to what

extent is the singularity structure of the Calabi-Yau variety captured by the geometry of the surface. In fact, one must retain more information than the intrinsic data of the two dimensional toric variety represented by the toric fan. We choose to retain this extra embedding information by using the language of stacks. We will review some elementary aspects of *quotient stacks* and *toric stacks* shortly.

We could avoid stacks altogether, but they give a very convenient setting in which to do our calculations. We emphasize that stacks are useful because of our $2d$ perspective, but could be avoided if we kept a $3d$ description of the singularity.

Without proofs or motivation, let us sketch a few of the, from our point of view, relevant aspects of quotient stacks. For a nice review we refer to [42]. Let X be an algebraic variety, and G be a group acting on it. Giving a coherent sheaf F on the quotient stack $[X/G]$ is equivalent to giving a coherent sheaf $\mathcal{F}$ on X, together with a G-equivariant structure. The G-equivariant sheaves do not in general descend to the quotient space X/G. The sheaves on X/G are precisely those G-equivariant sheaves, $\mathcal{F}$, for which the stabilizer G_x of any point $x \in X$ acts trivially on the stalk $\mathcal{F}_x$. In this sense, stacks allow for more general sheaves on the "quotient space" than varieties; we exploit this excess to write down our exceptional collections.

For D-branes and quivers, our bread and butter is computing Ext groups. Moving into the stacky context, one has a very simple result. Let E and F be two coherent sheaves on the stack $[X/G]$. If we represent them using two G-equivariant sheaves $\mathcal{E}$ and $\mathcal{F}$, then the stacky Ext is given by the invariant part of the "old-fashioned" Ext:

$$\mathrm{Ext}^n_{[X/G]}(\mathsf{E},\mathsf{F}) \,=\, \mathrm{Ext}^n_X(\mathcal{E},\mathcal{F})^G \,. \tag{6.1}$$

In recent years the stringy orbifold cohomology has been understood mathematically [43]. An algebraic version of this theory has been defined in the context of stacks [44]. More recently the orbifold Chow ring of a general toric Deligne-Mumford stack was obtained by Borisov, Chen and Smith (BCS) [19]. The BCS construction is completely combinatorial. In particular, Borisov, Chen and Smith give a procedure for associating a toric stack to a finitely generated abelian group N (rather than a lattice), a fan Σ in the free part of the group (i.e., in the vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$), and a collection of group elements (these are points on the rays of the fan with decorations that are valued in the finite part of the abelian group). We will use the the BCS results in this section, and we refer for details of the construction and references to [19, 45].

Let us look at the simplest non-trivial example of interest to us, which in fact will appear in the sequel: the weighted projective line $\mathbb{P}^1(a,b)$, for a and b *not* necessarily relatively prime integers. As a variety, or scheme, $\mathbb{P}^1(a,b)$ is isomorphic to $\mathbb{P}^1$, as we already discussed in section 4.3. We will consider the stack $\mathbf{P}^1(a,b)$ from two different points of view: first as a toric stack, and then as a quotient stack.

Torically it is somewhat inconvenient to talk about the variety $\mathbb{P}^1(a,b)$: once the lattice $\mathbb{Z}$ is given, the toric divisors are associated to the rays (the positive and negative axis), and not the non-minimal lattice points $-b$ and a on the rays:
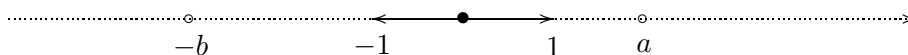
The non-minimal lattice points, $-b$ and a, correspond to the weights. Once we turn to stacks, in the language of toric Deligne-Mumford stacks [19], one can add extra decorations to each ray. In our case these are two elements r_1 and $r_2 \in \mathbb{Z}_k$, for a given integer k. In Kawamata's language [24] the decorations give rise to a boundary divisor $\frac{r_1-1}{r_1}D_1 + \frac{r_2-1}{r_2}D_2$. In particular, if $d = \gcd(a,b) > 1$, then we can choose $k = d$, and think of $\mathbb{P}^1(a,b)$ as $\mathbb{P}^1(\frac{a}{d}, \frac{b}{d})$ with $\mathbb{Z}_d$-valued decorations. In the BCS language this example is $N = \mathbb{Z} \oplus \mathbb{Z}_d$, the same fan as that of $\mathbb{P}^1$, and the choice of the two elements in N being $(-b, r_1)$ and (a, r_2).

Now let's see how $\mathbf{P}^1(a,b)$ is described as a quotient stack. In fact it is also true that $\mathbb{P}^1(a,b) \cong \mathbb{P}^1 / \mathbb{Z}_a \times \mathbb{Z}_b$, where the group $\mathbb{Z}_a \times \mathbb{Z}_b$ acts on the homogenous coordinates $(x_0 : x_1)$ of $\mathbb{P}^1$ as $(\omega_a x_0 : \omega_b x_1)$ — the ω's are the appropriate roots of unity. But once again, the $\mathbb{Z}_a \times \mathbb{Z}_b$ quotient is trivial on the variety. We can instead consider the stack $\mathcal{X} = \mathbf{P}^1(a,b) \cong [\mathbb{P}^1 / \mathbb{Z}_a \times \mathbb{Z}_b]$. This replaces the local quotient singularities of the weighted projective space $\mathbb{P}^1(a,b)$ by stacky structures in the weighted projective stack $\mathbf{P}^1(a,b)$.

From the above discussion, the stack $\mathcal{X} = [\mathbb{P}^1 / \mathbb{Z}_a \times \mathbb{Z}_b]$ has a richer category of coherent sheaves than the toric variety $\mathbb{P}^1 / \mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{P}^1$. To be more precise let $\pi \colon \mathcal{X} \to X$ denote the canonical projection from the stack to the space. Let $\bar{D}_1$ and $\bar{D}_2$ denote the two toric divisors of $X = \mathbb{P}^1(a,b)$, i.e., $x_1 = 0$ resp. $x_2 = 0$. They are of course linearly equivalent on $X = \mathbb{P}^1$. As explained in [24], the map $\pi \colon \mathcal{X} \to X$ ramifies along the D_i's, and there are divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{X}$ such that $\pi^*\bar{D}_i = s_i \mathcal{D}_i$. The s_i's here correspond to the r_i's of the toric BCS language.

But there is another covering map, since we are quotienting $\mathbb{P}^1$ by $\mathbb{Z}_a \times \mathbb{Z}_b$: $\mathbb{P}^1 \to \mathbf{P}^1(a,b)$ (see [33] for more details). We can summarize the two covering maps in the commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1(a,b) \\
& \searrow_{\varpi} \quad \nearrow_{\pi} & \\
& \mathbf{P}^1(a,b) &
\end{array}
\tag{6.2}
$$

If D_1 and D_2 denote the two toric divisors of $\mathbb{P}^1$, i.e., $D_i = \mathrm{div}(x_i)$, then $\varpi(D_i) = \mathcal{D}_i$ are divisors on $\mathcal{X} = \mathbf{P}^1(a,b)$. *But* $\varpi^*\mathcal{D}_1 = aD_1$ and $\varpi^*\mathcal{D}_2 = bD_2$. Nevertheless, there is an invertible sheaf $\mathcal{O}_\mathcal{X}(\mathcal{H})$ on $\mathcal{X} = \mathbf{P}^1(a,b)$ such that $\varpi^*\mathcal{O}_\mathcal{X}(\mathcal{H}) = \mathcal{O}_{\mathbb{P}^1}(1)$. This is the sense in which there are more sheaves on $\mathbf{P}^1(a,b)$ than on $\mathbb{P}^1$.

In the light of (6.1) the cohomology groups of $\mathcal{O}_\mathcal{X}(m\mathcal{H})$ are particularly easy to describe in the quotient stack language: the elements of $\mathrm{H}^0(\mathcal{X}, \mathcal{O}_\mathcal{X}(m\mathcal{H}))$ are degree m polynomials in two variables x_0 and x_1, invariant under the $\mathbb{Z}_a \times \mathbb{Z}_b$ action. The group action is $(x_0 : x_1) \mapsto (\omega_a x_0 : \omega_b x_1)$. In other words, these are the total degree m polynomials in x_0^a and x_1^b. This result is very familiar; it is a trivial generalization of Cox's homogenous coordinate ring, which naively would not make sense for a and b having a common factor. In this context x_0^a and x_1^b play the role of the variables to which we would normally assign degrees a resp. b. We can conveniently assemble this discussion into a single equation

$$
\bigoplus_{m \in \mathbb{N}} \mathrm{H}^0(\mathcal{X}, \mathcal{O}_\mathcal{X}(m\mathcal{H})) = \mathbb{C}[u,v],
\tag{6.3}
$$

where $\mathbb{C}[u,v]$ is a $\mathbb{Z}$-graded polynomial ring with $\deg u = a$ and $\deg v = b$.

Although our example was almost trivial, there is much more to toric stacks than meets the eye. We need and use these stacks in our proofs. We start out by proving that the conjectured collection for $Y^{p,p-1}$ is strongly exceptional for all p. Next we prove the same thing for any $Y^{p,q}$ space for which $p - q$ is even, say $p - q = 2r$, and $\gcd(p, r) = 1$. Finally, we extend these results to the corresponding $X^{p,q}$ spaces.

6.1 $Y^{p,p-1}$

Let us start with the collection of line bundles listed in eq. (5.13). Using linear equivalence we can bring it to a form that is more suitable for proving that it is strongly exceptional for all p. The toric divisors of $Y^{p,p-1}$ satisfy the following relations:

$$D_4 \sim D_2, \quad D_1 + D_2 \sim (p-1)D_3 \tag{6.4}$$

Using these linear equivalences, the collection (5.13) becomes

$$\mathcal{O}_X, \ \mathcal{O}_X(D_1), \ \mathcal{O}_X(D_3), \ \mathcal{O}_X(2D_3), \ \dots, \mathcal{O}_X((2p-2)D_3). \tag{6.5}$$

Now we should recall from section 5.2.2 the basic geometry of the $Y^{p,p-1}$ spaces. With our choice of the origin in the $x_1 = 1$ plane, $Y^{p,p-1}$ is the total space of the canonical sheaf over a complex surface $X_{p,p-1}$, which is the blow-up of the smooth point on the weighted projective space $Y_{p,p-1} = \mathbb{P}(p-1, 1, p-1)$.[11] Let $\pi\colon X_{p,p-1} \to Y_{p,p-1}$ be the blow-up map, x be the point that is blown up, and let $E = D_1$ be the exceptional divisor of the blow-up. Since x is a smooth point we have that $E^2 = -1$, and furthermore, in this case, $E = \mathbb{P}^1$.

The reader is now fully aware of the subtlety in handling $Y_{p,p-1} = \mathbb{P}(p-1, 1, p-1)$, and the previous paragraph as it stands is incomplete. We will use the fact that $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}((2p-2))$ is a strongly exceptional collection on the stack $\mathbf{P}(p-1, 1, p-1)$. Of course this statement would fail for the variety $\mathbb{P}(p-1, 1, p-1) \cong \mathbb{P}^2$. If we denote the minimal toric divisors of $\mathbb{P}(p-1, 1, p-1)$ by D_i, then the natural map $\pi\colon \mathbf{P}(p-1, 1, p-1) \to \mathbb{P}(p-1, 1, p-1)$ ramifies along D_2. The blow-up on the other hand took place in the atlas corresponding to the cone formed by D_1 and D_3. This mismatch, or avoidance if you will, allows us to use both classical algebraic geometry and the stacky exceptional collection. As we already said, one could forget about stacks, and instead keep track of the $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$-equivariant structure, but we find it more convenient to work in this hybrid setting. In particular, the theorems of classical algebraic geometry still apply in this subsection.

Now we are in a position to state the main result of this subsection:

Proposition 6.1. *The collection of rank 1 sheaves* $\mathcal{O}_X, \mathcal{O}_X(D_1), \mathcal{O}_X(D_3), \mathcal{O}_X(2D_3), \dots,$ $\mathcal{O}_X((2p-2)D_3)$, *on the complex surface* $X = X_{p,p-1}$ *is strongly exceptional.*

Proof. We prove the proposition by computing the Ext groups between any two elements in the collection, including any element with itself. We start with $\mathcal{O}_X$ and $\mathcal{O}_X(D_1)$,[12] and use a direct method. For the rest of the collection we derive some general results, which will be of use later on as well.

---

[11] The reader should *not* confuse $Y^{p,p-1}$ with $Y_{p,p-1}$. The former is a 3-fold, while the latter is a surface.

[12] We drop the subscript $(p, p-1)$ from $X_{p,p-1}$, and similarly for $Y_{p,p-1}$.

To start, we consider the short exact sequence (SES)[13]

$$0 \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0 . \tag{6.6}$$

The associated cohomology long exact sequence (LES) implies that $H^i(X, \mathcal{O}_X(-E)) = 0$, for $i \geq 0$.

Tensoring by the locally free, hence flat, sheaf $\mathcal{O}_X(E)$ the SES (6.6) gives another SES

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_E(-1) \longrightarrow 0 , \tag{6.7}$$

where we used the fact that $E^2 = -1$. Now $\mathcal{O}_{\mathbb{P}}^1(-1)$ is a remarkable sheaf in that $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}^1(-1)) = 0$, for all i. Therefore the associated cohomology LES implies that $H^0(X, \mathcal{O}_X(E)) = H^0(X, \mathcal{O}_X) = \mathbb{C}$ and $H^i(X, \mathcal{O}_X(E)) = H^i(X, \mathcal{O}_X) = 0$, for $i \geq 1$.

Thus, so far, we have established that $\mathcal{O}_X$ and $\mathcal{O}_X(E)$ form a strongly exceptional collection. To compute the Ext's between the $\mathcal{O}_X(mD_3)$'s we exploit the fact that they arise from an exceptional collection on Y, via the pull-back $\pi\colon X \to Y$. More precisely we have that

$$\mathcal{O}_X(D_3) = \pi^* \mathcal{O}_Y(1), \quad \text{and consequently} \quad \mathcal{O}_X(mD_3) = \pi^* \mathcal{O}_Y(m) \quad \text{for any } m \in \mathbb{Z} . \tag{6.8}$$

The first question is therefore to compute $\mathrm{Ext}_X^i(\pi^* \mathcal{O}_Y(a), \pi^* \mathcal{O}_Y(b))$, for $a, b \in \mathbb{Z}$. We will prove a Parseval-type identity

Lemma 6.2. $\mathrm{Ext}_X^i(\pi^* \mathcal{O}_Y(a), \pi^* \mathcal{O}_Y(b)) = \mathrm{Ext}_Y^i(\mathcal{O}_Y(a), \mathcal{O}_Y(b))$, *for all* $a, b \in \mathbb{Z}$.

Proof of the Lemma. The local freeness of the $\pi^* \mathcal{O}_Y(a)$'s reduces the problem to computing $H^i(X, \pi^* \mathcal{F})$, where $\mathcal{F}$ is a line bundle. For this we can use the ubiquitous *Leray spectral sequence*, which for a map $f\colon X \to Y$, and a sheaf $\mathcal{E}$ on X reads:

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{E}) \Longrightarrow H^{p+q}(X, \mathcal{E}) . \tag{6.9}$$

For us $\mathcal{E} = \pi^* \mathcal{F}$, and the Leray spectral sequence becomes

$$E_2^{p,q} = H^p(Y, R^q \pi_* \pi^* \mathcal{F}) \Longrightarrow H^{p+q}(X, \pi^* \mathcal{F}) . \tag{6.10}$$

But we can trivially rewrite $R^q \pi_* \pi^* \mathcal{F}$ as $R^q \pi_* (\pi^* \mathcal{F} \otimes \mathcal{O}_X)$. Now we can use the projection formula[14] since our $\mathcal{F}$ is a line bundle and get $\mathcal{F} \otimes R^q \pi_* \mathcal{O}_X$. But π is a smooth blow-up of a smooth surface, and Proposition 3.5 in section V.3 of [46] tells us that $R^q \pi_* \mathcal{O}_X = \mathcal{O}_Y$ for $q = 0$ and 0 otherwise. The same conclusion can be drawn using a general result about birational toric maps (see, e.g., page 76 of [22]). So the Leray spectral sequence (6.10) degenerates and $H^p(X, \pi^* \mathcal{F}) \cong H^p(Y, \mathcal{F})$, as promised. □

---

[13]We abbreviate "short exact sequence" as SES, and "long exact sequence" as LES.

[14]For a morphism of schemes $f\colon X \to Y$, a coherent sheaf $\mathcal{F}$ on X, and a locally free sheaf $\mathcal{E}$ on Y, the projection formula is the isomorphism $R^i f_* (\mathcal{F} \otimes f^* \mathcal{E}) \cong R^i f_* \mathcal{F} \otimes \mathcal{E}$. (See, e.g., III.9 of [46].)

Using the lemma, and the fact that the standard exceptional collection $\langle \mathcal{O}_Y, \mathcal{O}_Y(1), \ldots,$ $\mathcal{O}_Y(2p-2)\rangle$ on $Y = \mathbb{P}^2(p-1, 1, p-1)$ is strong, we can immediately conclude that $\langle \mathcal{O}_X, \mathcal{O}_X(D_3), \ldots, \mathcal{O}_X((2p-2)D_3)\rangle$ is a strong exceptional collection on X.

Finally we turn our attention to gluing the two strong exceptional collections on X: $\langle \mathcal{O}_X, \mathcal{O}_X(E)\rangle$ and $\langle \mathcal{O}_X, \mathcal{O}_X(D_3), \ldots, \mathcal{O}_X((2p-2)D_3)\rangle$. Therefore we need to evaluate the $\mathrm{Ext}_X^i(\pi^*\mathcal{O}_Y(a), \mathcal{O}_X(E))$'s, which boils down to computing $\mathrm{H}^i(X, \mathcal{O}_X(\pm E + mH))$ where for brevity we denoted D_3 by H. This notation also reminds us of the fact that D_3 originates from the hyperplane divisor on $Y = \mathbb{P}^2(p-1, 1, p-1)$ and inherits certain properties.

Again we use some convenient SES's. First, tensoring (6.6) by $\mathcal{O}_X(mH)$ gives[15]

$$0 \longrightarrow \mathcal{O}_X(mH - E) \longrightarrow \mathcal{O}_X(mH) \longrightarrow \mathcal{O}_E \longrightarrow 0 . \qquad (6.11)$$

From the associated cohomology LES $\dim \mathrm{H}^0(X, \mathcal{O}_X(mH - E)) = \dim \mathrm{H}^0(X, \mathcal{O}_X(mH)) - 1$,[16] while $\mathrm{H}^i(X, \mathcal{O}_X(mH - E)) = \mathrm{H}^i(X, \mathcal{O}_X(mH))$ for $i > 0$. By the lemma above $\mathrm{H}^i(X, \mathcal{O}_X(mH)) = \mathrm{H}^i(Y, \mathcal{O}_Y(mH))$. The strong exceptional property of the collection on Y guarantees that the latter one is 0 for any $m \geq 0$ and $i > 0$.

Finally, tensoring (6.7) by $\mathcal{O}_X(mH)$ gives

$$0 \longrightarrow \mathcal{O}_X(mH) \longrightarrow \mathcal{O}_X(mH + E) \longrightarrow \mathcal{O}_E(-1) \longrightarrow 0 . \qquad (6.12)$$

In cohomology this translates into $\mathrm{H}^i(X, \mathcal{O}_X(mH + E)) = \mathrm{H}^i(X, \mathcal{O}_X(mH))$ for all $i \geq 0$. By the lemma, and the strong exceptionality on Y, these groups are all 0 for any $-(2p-1) \leq m < 0$. Some bookkeeping now shows that this in fact completes the proof. $\qquad\square$

6.2 $Y^{p,p-2}$ and $Y^{p,p-2r}$, for $\gcd(p, r) = 1$

It is natural to ask the question of when do the four points

$$V_1 = (1, 0, 0), \quad V_2 = (1, 1, 0), \quad V_3 = (1, p, p), \quad V_4 = (1, p - q - 1, p - q) \qquad (6.13)$$

from eq. (5.7), defining the $Y^{p,q}$ spaces, give rise to a surface that is a direct product of two curves. From the "topology" of the pointset it is immediate that a necessary and sufficient condition is to have the line connecting V_1 and V_3 intersect the line connecting V_2 and V_4 in a *lattice point*. It is elementary to see that the two lines intersect at $(\frac{p-q}{2}, \frac{p-q}{2})$, hence we require that

$$p - q = 2r, \qquad \text{for some } r \in \mathbb{N}. \qquad (6.14)$$

Choosing $(\frac{p-q}{2}, \frac{p-q}{2}) = (r, r)$ as the origin of the 2d fan we have the four vectors $v_1 = (-r, -r)$, $v_2 = (1 - r, -r)$, $v_3 = (p - r, p - r)$ and $v_4 = (r - 1, r)$. We depicted these four points and connecting lines in figure 6.

From the toric fan in figure 6 it is now intuitively clear that v_1 and v_3 form a weighted projective space $\mathbb{P}^1(p - r, r)$, while v_2 and v_4 form the projective space $\mathbb{P}^1$.[17] This fact can

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[15]Here we used again a projection-type formula, but now for the intersection of divisors on a smooth surface: $E \cdot H = E \cdot \pi^* L = \pi_* E \cdot L = \text{point} \cdot L = 0$, where L is the hyperplane divisor on Y, i.e., $\mathcal{O}_Y(1) = \mathcal{O}_Y(L)$.

[16]Negative dimension means trivial vector space.

[17]Note that there is no lattice point on the ray connecting the origin to $v_2 = (1 - r, -r)$ closer to the origin.
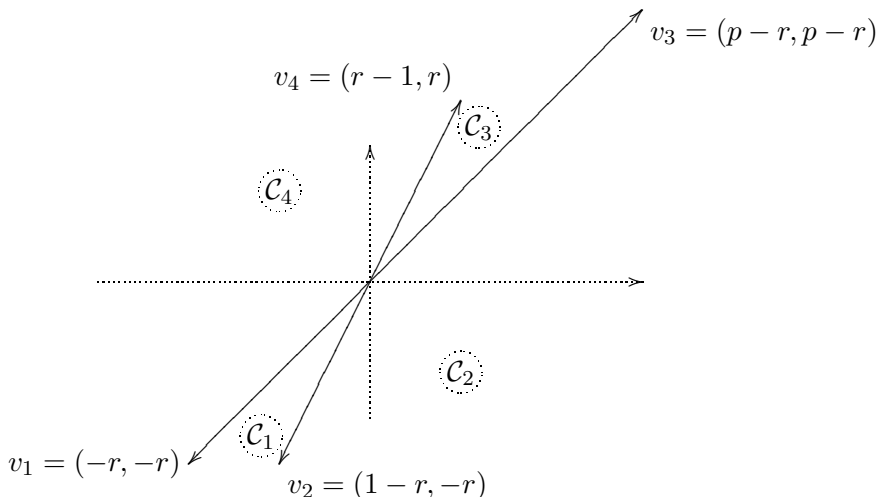
Figure 6: The toric fan of $Y^{p,q}$, for $p - q = 2r$.

be see even more explicitly if we use Cox's holomorphic quotient construction,[18] and once again we will have to deal with some stacky subtleties.

Let us work this out in detail. The space $Y^{p,q}$ is the total space of the canonical sheaf over a complex surface $X_{p,q}$. Figure 6 gives the toric fan of $X_{p,q}$. The linear equivalence relations among the toric divisors of $X_{p,q}$ are

$$D_4 \sim D_2, \quad rD_1 \sim (p-r)D_3 \,. \tag{6.15}$$

Let $x_1, \ldots, x_4$ be coordinates on $\mathbb{C}^4$. Following Cox [47], $X_{p,q}$ is the quotient of $\mathbb{C}^4 - \{x_1 = x_3 = 0, x_2 = x_4 = 0\}$ by $\mathbb{C}^* \times \mathbb{C}^*$, where the weights of the two $\mathbb{C}^*$ actions can be read out from (6.15): $(\lambda_1^{p-r} x_1, \lambda_2 x_2, \lambda_1^r x_3, \lambda_2 x_4)$, where $(\lambda_1, \lambda_2) \in \mathbb{C}^* \times \mathbb{C}^*$.

Consider for a moment the $x_1 = 0$ subspace, call it $\mathcal{D}_1$. Since the subset $\{x_1 = x_3 = 0\}$ is excluded and $x_1 = 0$ we must have $x_3 \neq 0$. Therefore we can use the first $\mathbb{C}^*$ action to completely fix x_3, say to $x_3 = 1$. What we are left with is $\mathbb{C}^2 - \{x_2 = x_4 = 0\}/\mathbb{C}^* \cong \mathbb{P}^1$. But we ignored the fact that $\mathbb{C}^*$ acts on x_3 with weight r. So we have to be more careful here, and see if there is any stacky subtlety.

To determine the "topology" of $\mathcal{D}_1$ as a stack we can use the toric method of [19]. This is actually interesting in itself, since we start with a toric variety with $N = \mathbb{Z}^2$ and no torsion, but for the subspace $\mathcal{D}_1$ the corresponding $N_{\mathcal{D}_1}$ will have torsion.

More generally, if we consider the abelian group $\mathbb{Z}^2$, and the subgroup $\langle (a, b) \rangle$ generated by (a, b), for $a, b \in \mathbb{Z}$, then the quotient $\mathbb{Z}^2 / \langle (a, b) \rangle$ is again a finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups it is necessarily of the form $\mathbb{Z}^r \oplus$ finite torsion. In fact it is easy to show that $\mathbb{Z}^2 / \langle (a, b) \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_{\gcd(a,b)}$. Using this it is automatic that $N_{\mathcal{D}_1} \cong \mathbb{Z} \oplus \mathbb{Z}_r$. The *Star* of $\mathcal{D}_1$ consists of the cones $\mathcal{C}_1$ and $\mathcal{C}_4$. The image of $v_2 = (1 - r, -r)$ in $N_{\mathcal{D}_1} \cong \mathbb{Z} \oplus \mathbb{Z}_r$ is $(1 - r, -r) \equiv (1, 0) \mod v_1$. Similarly, the image of $v_2 = (r - 1, r)$ in $N_{\mathcal{D}_1} \cong \mathbb{Z} \oplus \mathbb{Z}_r$ is $(-1, 0)$. Running the BCS construction one obtains a $\mathbb{P}^1$, with trivial $\mathbb{Z}_r$ action.

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[18]See appendix A.1 for a brief review.

The fact that $\mathcal{D}_1 = \mathbb{P}^1$ is only intuitively possible to be seen from Cox's construction, since the latter one gives a variety, rather than a stack. A similar analysis applies to the $x_3 = 0$ subspace, and we get another trivially-stacky $\mathbb{P}^1$.

The $x_2 = 0$ and $x_4 = 0$ subspaces are more straightforward. Consider the subspace $\mathcal{D}_2$ given by $x_2 = 0$. Once again, the $\{x_2 = x_4 = 0\}$ subset is excluded, and the second $\mathbb{C}^*$ action completely fixes x_4, yielding $\mathbb{C}^2 - \{x_1 = x_3 = 0\}/\mathbb{C}^*$, which is now isomorphic to $\mathbb{P}^1(p - r, r)$. The same result can be established by the toric stack method: now $v_2 = (1 - r, -r)$, but $\gcd(1 - r, -r) = 1$ for any r, and therefore $N_{\mathcal{D}_2} \cong \mathbb{Z}$. Thus in this case there is no room for any non-trivial extra group actions, but the naive weighs $(p - r, r)$ could be changed due to the quotienting. But this doesn't happen.

The *Star* of $\mathcal{D}_2$ now consists of the cones $\mathcal{C}_1$ and $\mathcal{C}_2$. Let's see what the lattice relation between v_1 and v_3 modulo v_2 is. Assume that $av_1 + bv_3 \equiv 0 \mod v_2$. Let us look at the 2nd component: $-ar + b(p - r) \equiv 0 \mod r$, which is equivalent to $b\,p \equiv 0 \mod r$. If we assume that $\gcd(p, r) = 1$, then b must divide r. This then shows that the relation satisfied in $N_{\mathcal{D}_2} \cong \mathbb{Z}$ by v_1 and v_3 is the one inherited from $N = \mathbb{Z}^2$, i.e., $(p - r)v_1 + rv_3 = 0$. We are not going to consider the case $\gcd(p, r) > 1$ for reasons to be explained shortly.

Since the two $\mathbb{C}^*$ actions are independent of each other, $X_{p,q}$ is the direct product of $\mathbb{P}^1$ and $\mathbb{P}^1(p - r, r)$: $X_{p,q} \cong \mathbb{P}^1 \times \mathbb{P}^1(p - r, r)$. The same conclusion can be drawn from the intersection relations $D_1 \cdot D_3 = 0$ $D_2 \cdot D_4 = 0$ and the linear equivalences (6.15). Let us summarize this in the following diagram, where p_1 and p_2 denote the projections:

$$
\begin{array}{ccc}
 & X_{p,q} & \\
 {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\
 \mathbb{P}^1 & & \mathbb{P}^1(p - r, r)\,.
\end{array}
\tag{6.16}
$$

Both $\mathbb{P}^1$ and $\mathbb{P}^1(p - r, r)$ have strongly exceptional collections. One might hope that the *exterior tensor product* of the exceptional collections would be a strongly exceptional collection on $X_{p,q}$. On the other hand, if $p - r$ and r have a common divisor, than the obvious strongly exceptional collection on $\mathbb{P}^1(p - r, r)$ is shorter than p terms. This would result in an exceptional collection shorter than $2p$ terms on the associated $Y^{p,q}$ space. Using the techniques discussed in section 5.2.1, we can produce a length p exceptional collection on $\mathbb{P}^1(p - r, r)$, but that involves torsion Cartier divisors, which are harder to handle.[19] To avoid this, from now on we assume that $\gcd(\mathbf{p - r, r}) = \mathbf{1}$, which is equivalent to $\gcd(\mathbf{p, r}) = \mathbf{1}$.

The $p - q = 2r$ family contains the $Y^{p,p-2}$ spaces as a subfamily, with $r = 1$. The conjectured collection of line bundles for the $Y^{p,p-2}$ spaces is listed in eq. (5.28). Using the linear equivalences (6.15) we can conveniently rewrite this collection

$$
\begin{array}{l}
\mathcal{O}, \qquad \mathcal{O}(D_3), \qquad \mathcal{O}(2D_3), \qquad \ldots \qquad , \mathcal{O}((p - 1)D_3)\,, \\
\mathcal{O}(D_2), \mathcal{O}(D_2 + D_3), \mathcal{O}(D_2 + 2D_3), \ldots, \mathcal{O}(D_2 + (p - 1)D_3)\,.
\end{array}
\tag{6.17}
$$

But this collection is precisely, the suitably ordered, exterior tensor product of the exceptional collections on $\mathbb{P}^1$ and $\mathbb{P}^1(p - 1, 1)$, because of the isomorphisms: $\mathcal{O}(D_2) = p_1^* \mathcal{O}_{\mathbb{P}^1}(1)$

---

[19]We hope to return to this issue in a future publication.

and $\mathcal{O}(D_3) = p_2^* \mathcal{O}_{\mathbb{P}^1(p-1,1)}(1)$. Note that in the $r = 1$ case the second linear equivalence from (6.15) is simply $D_1 \sim (p-r)D_3$, and hence D_3 is a minimal divisor.

With a suitable modification the above statement in fact holds for all the $X_{p,q}$ spaces we considered in this section. For $r \geq 2$ the divisor D_3 is non-minimal, and we need to use a suitable replacement D_3', where $\mathcal{O}(D_3') = p_2^* \mathcal{O}_{\mathbb{P}^1(p-r,r)}(1)$, whereas $D_3 = rD_3'$. On the other hand $\mathcal{O}(D_2) = p_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ still holds. With this notation we have

Proposition 6.3. *The collection of rank 1 sheaves*

$$\mathcal{O}_X, \, \mathcal{O}_X(D_3'), \, \mathcal{O}_X(2D_3'), \, \dots, \mathcal{O}_X((p-1)D_3'),$$

$$\mathcal{O}_X(D_2), \, \mathcal{O}_X(D_2 + D_3'), \, \mathcal{O}_X(D_2 + 2D_3'), \, \dots, \mathcal{O}_X(D_2 + (p-1)D_3')$$

on the complex surface $X_{p,q}$ is strongly exceptional, for any p and q that satisfy $p - q = 2r$, for some $r \in \mathbb{N}$ and $\gcd(p,r) = 1$.

Proof. Once again the proof consists of a systematic computation of the Ext groups between two elements in the collection, including any element with itself. The techniques that we use are similar to the $Y^{p,p-1}$ case, but the details are very different: for $Y^{p,p-1}$ we were dealing with a blow-up, while here we have fibrations (although trivial).

We start by proving the fact that the collection of sheaves $\mathcal{O}_X, \, \mathcal{O}_X(D_3'), \, \mathcal{O}_X(2D_3'), \, \dots,$ $\mathcal{O}_X((p-1)D_3')$ is strongly exceptional. The issue can be reduced to a problem on $Y = \mathbb{P}^1(p-r,r)$. We can again prove a Parseval-type identity

Lemma 6.4. $\operatorname{Ext}_X^i(p_2^* \mathcal{O}_Y(a), p_2^* \mathcal{O}_Y(b)) = \operatorname{Ext}_Y^i(\mathcal{O}_Y(a), \mathcal{O}_Y(b))$, *for all* $a, b \in \mathbb{Z}$.

Proof of the Lemma. Since the $p_2^* \mathcal{O}_Y(a)$'s are invertible sheaves, the problem reduces to computing $\operatorname{H}^i(X, p^* \mathcal{F})$, where $\mathcal{F}$ is an invertible sheaf. To evaluate this, we can use the Leray spectral sequence:

$$E_2^{p,q} = \operatorname{H}^p(Y, \, R^q p_{2*} \, p_2^* \mathcal{F}) \Longrightarrow \operatorname{H}^{p+q}(X, \, p_2^* \mathcal{F}). \tag{6.18}$$

Once again, by the projection formula $R^q p_{2*} \, p^* \mathcal{F} = R^q p_{2*}(p_2^* \mathcal{F} \otimes \mathcal{O}_X) = \mathcal{F} \otimes R^q p_{2*} \mathcal{O}_X$. To compute $R^q p_{2*} \mathcal{O}_X$ we use Grauert's theorem (Theorem 12.8 and Corollary 12.9 of [46]), which shows that $R^q \pi_* \mathcal{O}_X = \mathcal{O}_Y$ for $q = 0$ and 0 otherwise.[20] So the Leray spectral sequence (6.10) degenerates and $\operatorname{H}^p(X, \pi^* \mathcal{F}) \cong \operatorname{H}^p(Y, \mathcal{F})$. $\square$

Using the lemma, and the fact that the standard exceptional collection $\langle \mathcal{O}_Y, \mathcal{O}_Y(1), \dots,$ $\mathcal{O}_Y(p-1) \rangle$ on $Y = \mathbb{P}(p-r,r)$ is strong, we conclude immediately that $\langle \mathcal{O}, \mathcal{O}(D_3'), \dots, \mathcal{O}((p-1)D_3') \rangle$ is a strong exceptional collection on X.

Next we deal with the exceptionality of $\mathcal{O}_X$ and $\mathcal{O}_X(\pm D_2)$. Considering the short exact sequence (SES)

$$0 \longrightarrow \mathcal{O}_X(-D_2) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{D_2} \longrightarrow 0, \tag{6.19}$$

the associated cohomology long exact sequence (LES) implies that $\operatorname{H}^i(X, \mathcal{O}_X(-D_2)) = 0$, for $i \geq 0$.

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[20]Invoking Grauert's theorem is somewhat of an overkill, but it does the job in a heartbeat.

Tensoring by $\mathcal{O}_X(D_2)$ the SES (6.19) gives another SES

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_{D_2} \longrightarrow 0 \, , \tag{6.20}$$

where we used the fact that the self-intersection $D_2 \cdot D_2 = 0$.[21] The associated cohomology LES implies that $\mathrm{H}^0(X, \mathcal{O}_X(D_2)) = \mathbb{C}^2$, and $\mathrm{H}^i(X, \mathcal{O}_X(D_2)) = 0$ for $i \geq 1$.

To conclude, we need to compute the Ext's between the $\mathcal{O}_X(mD_3')$'s and the $\mathcal{O}_X(D_2 + kD_3')$'s. This is equivalent to computing $\mathrm{H}^i(X, \mathcal{O}_X(\pm D_2 + mD_3'))$, for both positive and negative n.

First, tensoring (6.19) by $\mathcal{O}_X(mD_3')$ gives

$$0 \longrightarrow \mathcal{O}_X(mD_3' - D_2) \longrightarrow \mathcal{O}_X(mD_3') \longrightarrow \mathcal{O}_{D_2}(mD_3') \longrightarrow 0 \, . \tag{6.21}$$

Let us remind ourselves that in the above equation $\mathcal{O}_{D_2}(mD_3')$ really means $j_*\mathcal{O}_{D_2} \otimes \mathcal{O}_X(mD_3')$, where $j : D_2 \hookrightarrow X$ is the embedding, and in fact $D_2 \cong Y \cong \mathbb{P}^1(p-r, r)$. Therefore $j_*\mathcal{O}_{D_2} \otimes \mathcal{O}_X(mD_3') = j_*\mathcal{O}_{\mathbb{P}^1(p-r,r)}(m)$. As a consequence

$$\mathrm{H}^i(X, j_*\mathcal{O}_{\mathbb{P}^1(p-r,r)}(m)) \cong \mathrm{H}^i(\mathbb{P}^1(p-r, r), \mathcal{O}_{\mathbb{P}^1(p-r,r)}(m)) = \mathrm{H}^i(Y, \mathcal{O}_Y(m)) \, ,$$

which implies that

$$\mathrm{H}^i(X, \mathcal{O}_{D_2}(mD_3')) \cong \mathrm{H}^i(Y, \mathcal{O}_Y(m)) \, . \tag{6.22}$$

On the other hand, using the lemma, $\mathrm{H}^i(X, \mathcal{O}_X(mD_3') = \mathrm{H}^i(Y, \mathcal{O}_Y(m))$. Combining this with eq. (6.22) we have: $\mathrm{H}^i(X, \mathcal{O}_X(mD_3') = \mathrm{H}^i(X, \mathcal{O}_{D_2}(mD_3'))$. The associated cohomology LES of (6.21) then gives $\mathrm{H}^i(X, \mathcal{O}_X(mD_3' - D_2)) = 0$, for all i and all m.

Finally, tensoring (6.20) by $\mathcal{O}_X(mD_3')$ gives

$$0 \longrightarrow \mathcal{O}_X(mD_3') \longrightarrow \mathcal{O}_X(mD_3' + D_2) \longrightarrow \mathcal{O}_{D_2}(mD_3') \longrightarrow 0 \, . \tag{6.23}$$

By the lemma $\mathrm{H}^i(X, \mathcal{O}_X(mD_3')) = 0$ for all $i > 0$ and any $-(p-1) \leq m$. The cohomology LES then gives $\mathrm{H}^i(X, \mathcal{O}_X(mD_3' + D_2)) = 0$ for the same range of i and m. Which completes the proof. □

6.3 $X^{p,q}$

In section 5.4 we already discussed the fact that the $X^{p,q}$ space is in a sense a blow-up of the $Y^{p,q-1}$ space at a smooth point. More precisely, if $X_{p,q}$ denotes the surface whose canonical sheaf is the 3-fold $X^{p,q}$, and similarly for $Y^{p,q-1}$, then $X_{p,q}$ is a blow-up of $Y_{p,q-1}$ at a smooth point.

Given a collection on $Y_{p,q-1}$, one can immediately pull it back to $X_{p,q}$ using the blow-up map $\pi \colon X_{p,q} \to Y_{p,q-1}$, and try to augment it in a suitable way. We can in fact prove the following:

---

[21]The intersection ring of a toric stack is isomorphic to the intersection ring of the underlying variety (Lemma 5.1 of [19]). The orbifold Chow rings are very different though.

Proposition 6.5. *Let $\pi\colon X_{p,q} \to Y_{p,q-1}$ be the blow-up map. Given a strongly exceptional collection of invertible sheaves*

$$\mathcal{E} = \mathcal{O}_{Y_{p,q-1}}, E_2, E_3, \ldots E_n$$

on $Y_{p,q-1}$, the induced collection on $X_{p,q}$

$$\mathcal{E}' = \mathcal{O}_{X_{p,q}}, \mathcal{O}_{X_{p,q}}(D_5), \pi^* E_2, \pi^* E_3, \ldots, \pi^* E_n$$

is also strongly exceptional.

Whenever we know a strongly exceptional collection on $Y^{p,q-1}$, the proposition allows us to determine the gauge theory for the corresponding $X^{p,q}$ space.

Proof. Let $E = D_5$ be the exceptional divisor of the blow-up $\pi\colon X_{p,q} \to Y_{p,q-1}$. Since we blew up a smooth point we expect that $E^2 = -1$ and $E = \mathbb{P}^1$. These facts can be checked easily and explicitly by toric methods. First, there is a linear equivalence $D_2 \sim D_4 + D_5$. Second, the cones give the intersection products: $D_2 \cdot D_5 = 0$ and $D_4 \cdot D_5 = 1$. Then

$$E \cdot E = D_5 \cdot D_5 = D_5 \cdot (D_2 - D_4) = 0 - 1 = -1 \,. \tag{6.24}$$

To show that $E = \mathbb{P}^1$ we use the BCS method. Since $v_5 = (p - q - 2, p - q - 1)$, and $\gcd(p - q - 2, p - q - 1) = 1$, we have that $N_{D_5} = \mathbb{Z}$. The old relation $v_5 = v_4 + v_1$ now gives $v_4 + v_1 \cong 0 \mod v_5$. Since v_1 and v_4 are the *Star* of v_5 this shows that $E = \mathbb{P}^1$.

Now that $E^2 = -1$ and $E = \mathbb{P}^1$ we can follow step by step the proof of Proposition 6.1. To avoid repetition we will be sketchy on the details that are similar to the $Y^{p,p-1}$ case. Using eq. (6.6) and (6.7) an identical computation shows that $\mathcal{O}_X$ and $\mathcal{O}_X(E)$ form a strongly exceptional collection.[22]

To deal with the tail $\langle \pi^* E_2, \pi^* E_3, \ldots, \pi^* E_n \rangle$ we can use once again Lemma 6.2, and the assumption that $\mathcal{E} = \langle \mathcal{O}_{Y_{p,q-1}}, E_2, E_3, \ldots E_n \rangle$ is strongly exceptional on Y.

Finally we have to glue the two strong exceptional collections on X: $\langle \mathcal{O}_X, \mathcal{O}_X(E) \rangle$ and $\langle \pi^* E_2, \pi^* E_3, \ldots, \pi^* E_n \rangle$. This boils down to computing $\mathrm{H}^i(X, \pi^* E_m(\pm E))$, for $m \in \mathbb{Z}$, both positive and negative.

Let's do $\mathrm{H}^i(X, \pi^* E_m(-E))$ first. Tensoring (6.6) with $\pi^* E_m$ gives

$$0 \longrightarrow \pi^* E_m(-E) \longrightarrow \pi^* E_m \longrightarrow \mathcal{O}_E \otimes \pi^* E_m \longrightarrow 0 \,. \tag{6.25}$$

Now remember that $\mathcal{O}_E \otimes \pi^* E_m$ is in fact $j_* \mathcal{O}_E \otimes \pi^* E_m$, where $j\colon E \hookrightarrow X$ is the embedding. Using the projection formula

$$j_* \mathcal{O}_E \otimes \pi^* E_m = j_*(\mathcal{O}_E \otimes j^* \pi^* E_m) = j_*(\mathcal{O}_E \otimes (\pi \circ j)^* E_m) \,. \tag{6.26}$$

But the map $\pi \circ j\colon E \hookrightarrow Y$ projects E to a point, the one that was blown up. By assumption the E_m's are invertible sheaves, and hence $(\pi \circ j)^* E_m = \mathcal{O}_E$.

From the cohomology LES associated to (6.25) we have that $\dim \mathrm{H}^0(X, \pi^* E_m(-E)) = \dim \mathrm{H}^0(X, \pi^* E_m) - 1$, while $\mathrm{H}^i(X, \pi^* E_m(-E)) = \dim \mathrm{H}^i(X, \pi^* E_m)$ for $i > 0$. Using

---

[22]We drop the superscripts of $X_{p,q}$ and $Y_{p,q-1}$.

Lemma 6.2: $H^i(X, \pi^* E_m) = H^i(Y, E_m)$. The strong exceptional property of the collection on Y guarantees that the latter one is 0 for any $i > 0$.

Finally, tensoring (6.7) with $\pi^* E_m^\vee$, the dual of $\pi^* E_m$, gives

$$0 \longrightarrow \pi^* E_m^\vee \longrightarrow \pi^* E_m^\vee(E) \longrightarrow \mathcal{O}_E(-1) \otimes \pi^* E_m^\vee \longrightarrow 0 \,. \tag{6.27}$$

A computation identical to (6.26) shows that $j_* \mathcal{O}_E(-1) \otimes \pi^* E_m^\vee = j_* \mathcal{O}_E(-1)$. The cohomology LES associated to (6.27) then completes the proof. $\qquad\square$

7. Exceptional collection conjecture

Given a singular toric surface generated by four rays v_1, v_2, v_3 and v_4 such that the hull of the four rays is a convex quadrilateral, we conjecture a strongly exceptional collection of line bundles. To motivate this conjecture, we start with the Calabi-Yau cone, promoting our 2-dimensional vectors v_i to coplanar three vectors $V_i = (1, v_i)$. While the v_i will in general satisfy two relations, the V_i will in general satisfy only one such relation which we take to have the form

$$a V_1 - c V_2 + b V_3 - d V_4 = 0 \,, \tag{7.1}$$

where we choose the integers to satisfy the inequalities $0 < a \le c \le d \le b$. The Calabi-Yau condition means that $a + b = c + d$. (Note that the v_i also satisfy this relation (7.1).) From these weights $(a, -c, b, -d)$ we will construct a quiver gauge theory. From the quiver, we will extract a collection of line bundles which we believe form a complete and strongly exceptional collection.

We insist that $\gcd(a, b, c, d) = 1$ or our method will give several copies of the quiver for $(a, b, c, d)/\gcd(a, b, c, d)$. More precisely, we would like there to be no torsion divisors in the singular surface and so take the v_i to span the $\mathbb{Z}^2$ lattice.

To construct the quiver, we rely on the relation between bifundamental fields in the gauge theory and divisors. For these toric varieties the simplest divisors are the ones invariant under the torus-action, which are in one-to-one correspondence with the rays of the quiver. For each V_i, we have a T-invariant divisor D_i. From [37] the number of bifundamental fields corresponding to D_i which appear in the quiver gauge theory is equal to the determinant $\langle V_{i-1}, V_i, V_{i+1} \rangle$ where we take $V_5 = V_1$ and $V_0 = V_4$. The proof comes from considering how many torsion line bundles can be constructed from each such divisor.

If we choose the v_i to be

$$v_1 = (1, 0) \,, \quad v_2 = (ak, b) \,, \quad v_3 = (-al, c) \,, \quad v_4 = (0, 0) \tag{7.2}$$

where $ck + bl = 1$, then we find that there are b bifundamental fields of type D_1, d bifundamentals of type D_2, a bifundamentals of type D_3, and c bifundamentals of type D_4.

From our exceptional collection techniques we expect that each node in the quiver can be identified with a line bundle on the Calabi-Yau cone X. A line bundle on X will in general take the form $\mathcal{O}(\sum k_i D_i)$. Since there are only four rays generating X, all the D_i are linearly equivalent and we can write this line bundle in an equally descriptive way as

$$\mathcal{O}(\sum k_i D_i) \cong \mathcal{O}(a k_1 - c k_2 + b k_3 - d k_4) \,. \tag{7.3}$$

We can thus think of a collection of line bundles $\mathcal{E} = (E_1, E_2, \ldots, E_n)$ as a set of integers, $\mathcal{E} = (m_1, m_2, \ldots, m_n)$ (up to torsion).

To have a complete exceptional collection, we need one line bundle for each type of D-brane. This number of objects is just the Euler character χ of the Calabi-Yau cone, which is also the number of triangles in a complete triangulation of the base of the toric cone [22]. In our case, $\chi = a + b = c + d$ [37].

The differences of line bundles in the exceptional collection can be identified with sums of divisors. If the difference is one of the D_i, then we can place a bifundamental field between those two nodes because there is no simpler way of expressing D_i. Thus, our exceptional collection will consist of a set of integers $\mathcal{E} = (m_1, m_2, \ldots, m_{a+b})$, which satisfy the following condition: there are b pairs i, j, such that $m_j - m_i = a$, d pairs such that $m_j - m_i = c$, a pairs such that $m_j - m_i = b$, and c pairs such that $m_j - m_i = d$. There are in general many solutions to these constraints which we expect to be related by Seiberg duality. However, one solution which always seems to work is the following.

Consider the set

$$\mathcal{E} = (N + 1, N + 2, \ldots, N + a + b) \ . \tag{7.4}$$

We draw all possible arrows from E_i to E_{i+a}, from E_i to E_{i+b}, from E_{i+c} to E_i, and from E_{i+d} to E_i. We are almost done, but there is a subtlety.

There may also be some bifundamental fields of the form $D_i + D_j$. Consider $D_3 + D_4$ first. The charge of this divisor is $b - d$. If we have two nodes m_{i+b-d} and m_i, then the difference must correspond to the divisor $D_3 + D_4$. However, it is not always true that we can think of this map as composed individually of a D_3 map from m_j to m_{j+b} and a D_4 map from m_k to m_{k-d}. We find that for $a < i \leq d$, no such combination exists. So we need to add $d - a$ bifundamental fields corresponding to the divisor $D_3 + D_4$ for i in this range. A similar situation exists for $D_2 + D_3$. Here the charge is $b - c$. Now there will be no combination of maps that sum to $D_2 + D_3$ for $a < i \leq c$. In other words, we must add $c - a$ fields of type $D_2 + D_3$ that begin on nodes i for $a < i \leq c$. We don't need to draw any arrows for $D_1 + D_2$ or for $D_1 + D_4$ because for our $\mathcal{E}$, these divisors can be produced by combining maps corresponding to the individual D_i. A similar situation holds for $D_1 + D_3$, $D_2 + D_4$, and the $2D_i$.

Let us do an example. Consider $L^{2,6,3}$ which has the single relation

$$2V_1 - 3V_2 + 6V_3 - 5V_4 = 0 \ . \tag{7.5}$$

There will be eight nodes in the quiver ($6 + 2 = 8$), and eight objects in the collection

$$\mathcal{E} = (E_1, E_2, \ldots, E_8) \ . \tag{7.6}$$

We first draw in the arrows corresponding to the D_i. There will be six arrows from E_i to E_{i+2} and two arrows from E_i to E_{i+6}. In the opposite direction, there will be five arrows from E_i to E_{i-3} and three arrows from E_i to E_{i-5}. Finally we add in the arrows corresponding to the missing $D_2 + D_3$ and $D_3 + D_4$. There will be three arrows corresponding to $D_3 + D_4$, one from E_3 to E_4, one from E_4 to E_5 and one from E_5 to E_6. There will be one arrow corresponding to $D_2 + D_3$ from E_3 to E_6. This quiver, up to a reordering of the nodes, is exactly the same as figure 9 of [37].

Thus we have a simple recipe for computing a quiver given a toric Calabi-Yau cone generated by four rays. We also have the skeleton of an exceptional collection on a toric surface corresponding to the v_i. While the cone has only one relation, the toric surface will have two relations, one of which continues to be $(a, -c, b, -d)$. The toric surface, because it has an extra relation, will have

Figure 7: The gauge theory quiver for $L^{2,6,3}$. The dashed arrows correspond to the composite divisors $D_3 + D_4$ and $D_2 + D_3$.

one additional linearly independent divisor. Thus, one integer m_i is no longer enough to specify uniquely a line bundle. We need two integers: the set

$$\mathcal{O}_V(N+1, n_1), \mathcal{O}_V(N+2, n_2), \dots$$
$$\dots, \mathcal{O}_V(N+a+b, n_{a+b}) \tag{7.7}$$

appropriately reordered for some set of integers n_i and N, should be a complete, strongly exceptional collection on our toric surface.

Motivated by the smooth del Pezzo case where the sheaves are ordered by their slope (as reviewed in section 2.3), we would like to choose n_i to be the degree of the line bundle. Having found a set of n_i that are sandwiched between 0 and K^2, we conjecture that the collection will be strongly exceptional ordered by the n_i.

Given the quiver and the identification of arrows in the quiver with toric divisors D_i, we can momentarily disregard this skeleton collection, and build a new collection from scratch. Pick any node of the quiver and associate to it $\mathcal{O}_V$. Working outward from $\mathcal{O}_V$, given two nodes i and j and an arrow from i to j corresponding to a divisor D, we associate line bundles $\mathcal{O}_V(C_i)$ and $\mathcal{O}_V(C_j)$ to the nodes such that $C_i + D = C_j$. As there are many loops in the quiver, this procedure is not uniquely determined. Also note that because of these loops, we will always find nodes connected by an arrow such that $C_i + D \neq C_j$. However in this case, if we let

$$C_i + D - C_j = \sum k_i D_i \tag{7.8}$$

then, $ak_1 - ck_2 + bk_3 - dk_4 = 0$. In other words, as we can see from our skeleton collection, the discrepancy will be trivial when lifted to the Calabi-Yau cone.

At this point, we need to check whether all our line bundles have degree bounded between 0 and K^2. We take a brief detour to explain how to calculate $-K \cdot D$ from the fan. What we have to do is toric intersection theory on an orbifold, and is well known [22]. Given D_i and D_j, $i \neq j$, if v_i and v_j are not rays of the same maximal cone, then $D_i \cdot D_j = 0$. On the other hand, if v_i and v_j do belong to a maximal cone, then $D_i \cdot D_j$ is the reciprocal of the index of the sub-lattice generated by v_i and v_j in $\mathbb{Z}^2$. The index can be conveniently computed using a two-by-two determinant:

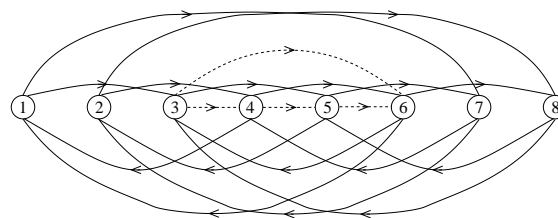$$D_i \cdot D_{i+1} = \frac{1}{|v_i, v_{i+1}|} .$$

For a fan generated by n rays, these $n(n-1)/2$ relations generally overdetermine the intersection matrix. The remaining D_i^2 can be determined from linear equivalence, picking two of the divisors and expressing them as linear combinations of the remaining $n-2$.

In this not uniquely determined procedure for labeling the nodes of the quiver, we search out the line bundle with the smallest and the line bundle with largest degree. If the difference in degree is more than K^2, then we take the largest degree line bundle and add K. We continue this procedure until all the line bundles are sandwiched between 0 and K^2.

Again, consider the example $L^{2,6,3}$. We take the fan of the toric surface to be

$$v_1 = (1,1) , \quad v_2 = (-1,0) , \quad v_3 = (0,-2) , \quad v_4 = (1,-2) . \tag{7.9}$$

We will call $E_3 = \mathcal{O}_V$. Then we find, in the wrong order suggested by the skeleton collection,

$$\begin{aligned} \mathcal{E} =& \mathcal{O}_V(3D_3 + 4D_4), \mathcal{O}_V(4D_3 + 5D_4), \mathcal{O}_V, \\ & \mathcal{O}_V(D_3 + D_4), \mathcal{O}_V(2D_3 + 2D_4), \ldots, \mathcal{O}_V(5D_3 + 5D_4) . \end{aligned} \tag{7.10}$$

From the fan, we find that

$$D_3 \cdot D_4 = \frac{1}{2} , \quad D_3^2 = -\frac{1}{2} , \quad D_4^2 = -\frac{1}{3} , \tag{7.11}$$

and, using linear equivalence, $-K = 5D_3 + 6D_4$. Thus, $-K \cdot (a_3 D_3 + a_4 D_4) = (a_3 + a_4)/2$. Thus, the correct order should be

$$\mathcal{E} = \mathcal{O}_V, \mathcal{O}_V(D_3 + D_4), \mathcal{O}_V(2D_3 + 2D_4), \mathcal{O}_V(3D_3 + 3D_4), \mathcal{O}_V(3D_3 + 4D_4),$$

$$\mathcal{O}_V(4D_3 + 4D_4), \mathcal{O}_V(4D_3 + 5D_4), \mathcal{O}_V(5D_3 + 5D_4) . \tag{7.12}$$

This collection is indeed sandwiched between 0 and $K^2 = 11/2$. In labeling the nodes, we could easily have chosen $\mathcal{O}_V(2D_1 + 2D_2) \sim \mathcal{O}_V(8D_3 + 10D_4)$ instead of $\mathcal{O}_V(3D_3 + 4D_4)$. After calculating the degrees, we would have found that the collection is not sandwiched between 0 and K^2 and would have known to add K to $8D_3 + 10D_4$.

Indeed, our computer routines verified that this collection is strongly exceptional in the order presented. We can then complete the circle and check that this collection also generates the correct quiver. The computer routines yield

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{7.13}$$

This matrix gives rise to precisely the quiver discussed above. There is one subtlety however. The quiver we discussed has a bidirectional arrow between what we have labeled nodes 1 and 4. We find $S_{14}^{-1} = 0$ and would have to study explicitly the inverse sheaves to detect these two maps from the collection.

Before concluding this section, we consider one more example, $Y^{3,1}$. As above, we take the v_i to satisfy the relation $4v_1 - 3v_2 + 2v_3 - 3v_4$. Before, we considered two distinct strongly exceptional collections for this space. There was the collection that gave rise to a quiver with two single impurities (5.32):

$$\mathcal{E} = \mathcal{O}_V, \mathcal{O}_V(D_4), \mathcal{O}_V(D_3 + D_4), \mathcal{O}_V(D_3 + 2D_4), \mathcal{O}_V(2D_3 + 2D_4), \mathcal{O}_V(2D_3 + 3D_4) \ . \quad (7.14)$$

Lifting this collection to the Calabi-Yau cone, we find

$$\mathcal{E} = (0, -3, -1, -4, -2, -5) \tag{7.15}$$

which is precisely of the form (7.4). We also found a Seiberg dual quiver with a single double impurity (5.21):

$$\mathcal{E} = \mathcal{O}_V, \mathcal{O}_V(D_3), \mathcal{O}_V(2D_3), \mathcal{O}_V(D_4), \mathcal{O}_V(D_3 + D_4), \mathcal{O}_V(2D_3 + D_4) \ . \tag{7.16}$$

Lifting this collection to the cone yields

$$\mathcal{E} = (0, 2, 4, -3, -1, 1) \tag{7.17}$$

which is not of the form (7.4). Nevertheless, one can check that there are still 2 fields of type D_3, 4 fields of type D_1, and 3 fields of both type D_2 and D_4, as required.

8. Discussion

What we have done in this paper is nontrivial and surprising. Through a combination of guesswork and deduction, we have found strongly exceptional collections on a large number of singular toric surfaces. Using some basic results in toric geometry and some more sophisticated results in local cohomology, we were able to prove that these collections were indeed strongly exceptional and from these collections derive quiver gauge theories for the D-branes probing the corresponding Calabi-Yau singularities.

A priori there were no guarantees this program would succeed. Although exceptional collections were known to exist on these singular surfaces [24], it was not clear that the more physical strongly exceptional collections could be found. It was also unclear that these singular surfaces — the result of only partially resolving the Calabi-Yau singularity — would contain the gauge theory information. It was not even clear that the mathematical technology existed to manipulate these collections once obtained. Indeed, some of the requisite stacky techniques were only developed in the last few years.

Although this paper is a good beginning, much remains to be done.

1. We need a general algorithm for converting a toric diagram into an exceptional collection. The techniques presented in this paper are quite powerful, and eventually we hope the exceptional collection method will be both the most rigorous and fastest way of converting geometry into gauge theory.

2. These $\mathcal{N} = 1$ supersymmetric gauge theories are specified by both a quiver and a superpotential. While we gave quivers, we did not derive superpotentials in this paper. Techniques exist for computing the superpotential from an exceptional collection in the smooth case [10, 58, 59], and we expect that these techniques generalize to the singular case as well.

3. We said very little about Seiberg duality. While we emphasized at the beginning that exceptional collections can produce conformal quiver gauge theories where the ranks of the gauge groups are not all equal, we only presented collections of line bundles where the ranks were equal. Seiberg duality will generically produce these higher rank gauge theories. Given the collections here, it should be straightforward to explore these equivalence classes of Seiberg dual theories.

4. Seiberg duality is defined as sheaf mutations only for well split nodes [25]. For the smooth del Pezzo case, this definition was ample because the physical exceptional collections all appear to be well-split. One burning question is how to define Seiberg duality for the ill-split $L^{p,q,r}$ examples discussed in the text.

5. Although we asserted that the bifundamental fields in the gauge theory come from the maps between the inverse collection, we never computed this inverse collection explicitly, just S^{-1}. The inverse collection, also often called a collection of fractional branes, lives in the derived category. In the smooth case, the fractional branes were one-term complexes. In the singular case one expects more complicated complexes. This has already been observed for the $\mathbb{C}^3/\mathbb{Z}_5$ orbifold [48], and it is true even in the $\mathbb{C}^2/\mathbb{Z}_n$ case for $n > 3$ [49].

6. There are a number of mathematical results for exceptional collections on del Pezzos that one would ideally hope to establish rigorously for the singular cases. One such result involves the existence of a helix. In the smooth case, one can prove that right mutating the E_1 in $\mathcal{E}$ to the end of the collection is equivalent to tensoring E_1 by the anti-canonical class. For the examples studied in the text this same result appears to hold in the singular case, but we lack a proof at the moment.

Acknowledgments

A. Computing sheaf valued toric cohomology

There are several useful ways to construct a toric variety X given its fan Δ. We will

mention only two of them.[23] Probably the most widely known method is to use the group algebra $\mathbb{C}[\sigma^{\vee}]$ of every dual cone $\sigma^{\vee}$, and patch the associated affine schemes together, as explained for example in [22]. In the case of projective spaces this construction is the analog of the affine coordinate charts.

For projective spaces on the other hand we also have the very efficient homogeneous coordinates at our disposal, and one can ask if there is an analog of this for toric varieties. This question was answered positively by Cox in 1995 [47]. Cox's construction is in terms of a GIT quotient, and is usually referred to as the *holomorphic quotient*. His construction generalizes the $\mathbb{C}^{n+1} - \{0\}/\mathbb{C}^*$ representation of $\mathbb{CP}^n$, replacing $\{0\}$ with a suitable subset and assigning certain weights to the different $\mathbb{C}^*$ generators. We will review elements of this construction shortly.

Our main interest is in computing sheaf cohomology groups for locally free sheaves, and in particular invertible ones, i.e., line bundles. The two toric variety constructions mentioned previously give two very different ways to achieve this. Computing $\mathrm{H}^0(X, \mathcal{O}(D))$ in the "old" affine-patching way boils down to counting lattice points inside a polytope associated to D. Higher cohomologies are harder to compute, but there is a well-known, if not very attractive, method [22]: one has to compute relative cohomology groups of simplicial topological spaces.

Another method to compute sheaf cohomology for torus equivariant bundles is popularized in [51], based on the work of Klyachko [52], and uses flags of vector spaces. For simple examples this is indeed efficient, but we found little practical use for it in our context.

Therefore it is natural to ask whether a cohomology computation based on the holomorphic quotient construction would be more convenient for practical purposes. This would have obvious advantages which extend beyond the scope of this paper, for example to the $(0, 2)$ gauged linear sigma models [53, 57]. Partly motivated by the authors of [57], Eisenbud, Mustata and Stillman tackled the problem of computing $\mathrm{H}^i(X, \mathcal{O}(D))$ in a uniform way using the toric homogeneous coordinate ring of Cox [17]. Since this work is virtually unknown in the physics community, we will review their approach in more detail. For the convenience of the reader we also follow their notation.

A.1 Gradings

We begin with Cox's homogeneous coordinate ring. Since this is well known, we only focus on some aspects that are relevant for the cohomology. Let Δ be the fan in $\mathbb{Z}^d$ corresponding to the toric variety X, and assume that Δ has n edges (one-dimensional cones). To every edge of Δ one associates a variable, and we get a polynomial ring $S := \mathbb{C}[x_1, \ldots, x_n]$. S has two obvious gradings:

1. $\mathbb{Z}$-grading: every variable is assigned degree 1;

2. $\mathbb{Z}^n$-grading: x_i has degree 1 under the ith $\mathbb{Z}$, and 0 under all other $\mathbb{Z}$'s, i.e., $\deg x_i = (0, \ldots, \overset{i\text{th}}{1}, \ldots, 0)$.

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[23]For a thorough discussion see [50], and references therein.

Cox introduces a new grading for S which is finer than the $\mathbb{Z}$-grading but coarser than the $\mathbb{Z}^n$-grading. Consider the matrix ρ, whose rows are the coordinates of the first integral points of the n edges of Δ.[24] After a choice of basis this matrix represents a linear map, and the torus-invariant divisor classes correspond to the elements of the cokernel $\mathbf{D}$ [22], yielding an exact sequence

$$\mathbb{Z}^d \xrightarrow{\ \rho\ } \mathbb{Z}^n \xrightarrow{\ \phi\ } \mathbf{D} \longrightarrow 0\,. \tag{A.1}$$

Now we introduce the grading that is coarser than the $\mathbb{Z}^n$-grading: the $\mathbf{D}$-grading.

For concreteness let us consider the mother of all examples, the projective space: $\mathbb{P}^n$. There are $n+1$ vertices $v_1 = (1, \ldots, 0), \ldots, v_n = (0, \ldots, 1), v_{n+1} = (-1, \ldots, -1) \in \mathbb{Z}^n$, and the fan has the obvious $n+1$ cones. The matrix ρ in (A.1) in this case becomes

$$\rho = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \tag{A.2}$$

with cokernel isomorphic to $\mathbb{Z}$. For future convenience call the generator of the cokernel Υ.

Any $\mathbb{Z}^n$-graded module N (and in particular S itself) can be regarded as a $\mathbf{D}$-graded module by setting

$$N_\delta = \bigoplus_{p \in \phi^{-1}\delta} N_p \quad \text{for each} \quad \delta \in \mathbf{D}\,. \tag{A.3}$$

In other words the ring $S := \mathbb{C}[x_1, \ldots, x_n]$ is graded by the divisor class group, and the degree of $x^D = \prod x_i^{a_i}$ equals

$$\deg(x^D) = \left[\sum a_i D_i \right] \in \mathrm{DivCl}(X)\,, \tag{A.4}$$

where D_i is the divisor associated to the ith edge. For our $\mathbb{P}^n$ we simply have $\deg(x_i) = 1 \cdot \Upsilon$, for all $i = 1, \ldots, n+1$. This is the usual assignment: the homogenous coordinates of $\mathbb{P}^n$ all have degree 1.

The *homogeneous coordinate ring* of X is the polynomial ring S together with the $\mathbf{D}$-grading and the *irrelevant ideal*

$$B = \left\langle \left\{ \prod_{v_i \notin \sigma} x_i \mid \sigma \in \Delta \right\} \right\rangle\,, \tag{A.5}$$

where σ ranges over the cones in the fan Δ. As an aside let us note that the zero set of the ideal B, $\mathbb{V}(B)$, is the exceptional set one discards in the GIT quotient and $\mathbf{D}$ gives rise to the $\mathbb{C}^*$ actions [47]. For our example $\mathbb{P}^n$, $B = \langle x_1, x_2, \ldots, x_{n+1} \rangle$ and $\mathbb{V}(B) = (0, 0, \ldots, 0)$, as expected.

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[24]Although the original paper [47] uses the minimal generators of the ray, the construction extends naturally to non-minimal generators.

The graded pieces of S have a well-known cohomological interpretation [47, 50]. If a Weil divisor $\mathbf{D}$ is not Cartier, then it determines a reflexive sheaf $\mathcal{O}_X(D)$, rather than a line bundle. Regardless of this, for any Weil divisor $\mathbf{D}$ we have an isomorphism:[25]

$$\mathrm{H}^0(X, \mathcal{O}_X(D)) = S_\delta, \quad \text{where } \delta = [D] \in \mathrm{DivCl}(X). \tag{A.6}$$

Therefore the 0th cohomology of an invertible sheaf can be computed from the knowledge of the $\mathbf{D}$-grading, which in turn is a purely algebraic Hilbert-function computation, and can be efficiently handled by "the machines".

One observes the conceptual similarity in the computation of H^0 between counting lattice points inside a polytope and counting monomials of a given degree. So what is the analog of the relative cohomology in the homogeneous coordinate context? The work of [17] extends this analogy to higher cohomologies and general toric sheaves (not only line bundles) in terms of Grothendieck's *local cohomology*. But before we can present this, we need some more background material.

Given a $\mathbf{D}$-graded S-module P, Cox [47] constructs a quasi-coherent sheaf $\tilde{P}$ on X by localizing as in the case of projective space. This is usually called the Serre functor. Coherent sheaves come from finitely generated modules:

Theorem A.1. (Theorem 2.1 in [17]) *Every coherent $\mathcal{O}_X$-module may be written as $\tilde{P}$ for a finitely generated $\mathbf{D}$-graded S-module P.*

For any $\mathbf{D}$-graded S-module P and any $\delta \in \mathbf{D}$ it is customary to define the *shifted* graded module $P(\delta)$, with the grading

$$P(\delta)_\epsilon = P_{\delta+\epsilon}, \quad \text{for all } \epsilon \in \mathbf{D}. \tag{A.7}$$

We introduce the following notation

$$\mathrm{H}^i_{\mathrm{tot}}(X, \tilde{P}) = \bigoplus_{\delta \in \mathbf{D}} \mathrm{H}^i(X, \widetilde{P(\delta)}). \tag{A.8}$$

In particular

$$\mathrm{H}^i_{\mathrm{tot}}(X, \mathcal{O}_X) = \bigoplus_{\delta \in \mathbf{D}} \mathrm{H}^i(X, \mathcal{O}_X(\delta)) \tag{A.9}$$

contains all the cohomology groups that we want. Note that by the above discussion (A.6) immediately implies that $\mathrm{H}^0_{\mathrm{tot}}(X, \mathcal{O}_X) \cong S$.

The advantage of working with $H^i_{\mathrm{tot}}(X, \tilde{P})$ is that it can be computed in terms of local cohomology, which in turn can be obtained as a limit of Ext modules. The first relevant result for us in this direction is an extension to toric varieties of a well-known projective result (see, e.g., appendix A4 of [40]):

Theorem A.2. (Prop. 2.3 in [17]) *Let P be a $\mathbf{D}$-graded S-module, $\tilde{P}$ be the corresponding quasi-coherent sheaf on X, and let $\mathrm{H}^i_B(P)$ denote the local cohomology module with support in B. Then we have:*

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[25]S_δ is the traditional way of denoting the collection of elements of degree δ.

(a) For $i \geq 1$, there is an isomorphism of graded S-modules

$$\mathrm{H}^i_{\mathrm{tot}}(X, \tilde{P}) \cong \mathrm{H}^{i+1}_B(P) \,. \tag{A.10}$$

(b) There is an exact sequence of graded S-modules

$$0 \longrightarrow \mathrm{H}^0_B(P) \longrightarrow P \longrightarrow \mathrm{H}^0_{\mathrm{tot}}(X, \tilde{P}) \longrightarrow \mathrm{H}^1_B(P) \longrightarrow 0 \,. \tag{A.11}$$

Let us spend a few moments elucidating the definition and intuition behind local cohomology. This will then illuminate the main result of [17], which we use for the explicit computations. For a concise introduction to local cohomology one can consult [40], while [54] provides a detailed treatment.

A.2 Local cohomology

Local cohomology was introduced by Grothendieck as an algebraic analog of the topological relative cohomology [55]. One can define local cohomology using a Čech complex, but instead we follow the abstract derived functor approach. To avoid confusion, let us add that local cohomology is in fact more general than sheaf cohomology, at least for projective schemes. So the name is a misnomer.

Let A be a ring, I be an ideal of A, and M be an A-module. The 0*th local cohomology module of M with support in I* is defined as the set of all elements of M that are annihilated by some power of I:

$$\Gamma^0_I(M) = \underset{n \in \mathbb{N}}{\cup} (0 :_M I^n) \,. \tag{A.12}$$

Here we used the standard notation $(0 :_M I^n) = \{m \in M \mid am = 0 \text{ for all } a \in I^n\}$. First, one observes the inclusion $(0 :_M I^n) \subset (0 :_M I^{n+1})$. Therefore the union in (A.12) is in fact a direct limit. Second, there is an isomorphism of A-modules $(0 :_M I^n) \cong \mathrm{Hom}_A(A/I^n, M)$.[26] Since I is an ideal, $I^{n+1} \subset I^n$, and thus $\mathrm{Hom}_A(A/I^n, M) \subset \mathrm{Hom}_A(A/I^{n+1}, M)$, exhibiting the earlier observed filtration. All this can be summarized in the form

$$\Gamma^0_I(M) = \lim_{n \to \infty} \mathrm{Hom}_A(A/I^n, M) \,. \tag{A.13}$$

There is a nice geometric interpretation of local cohomology. If we think of the elements of M as global sections of the sheaf $\tilde{M}$ on $\mathrm{Spec}\, A$, then the elements of $\Gamma^0_I(M)$ are the sections with support on the closed subscheme $\mathrm{Spec}\, A/I$.

One can show that $\Gamma^0_I(-)$ is a *left* exact functor. This allows us to define the higher local cohomology modules as its *right*-derived functors:[27]

$$\mathrm{H}^i_I(M) = R^i \Gamma^0_I(M) \,. \tag{A.14}$$

The useful thing for us is the fact that the representation (A.13) has a nice extension:

$$\mathrm{H}^i_I(M) \cong \lim_{n \to \infty} \mathrm{Ext}^i_A(A/I^n, M) \,, \tag{A.15}$$

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[26]Given $m \in (0 :_M I^n)$ one can define an A-module homomorphism $\varphi_m \colon A/I^n \to M$ by sending the class $a + I^n$ to am. Conversely, given an A-module homomorphism $\phi \colon A/I^n \to M$ one has $\phi(1) \in (0 :_M I^n)$.

[27]One uses an injective resolution for M.

where the isomorphism is natural, i.e., functorial. The authors of [17] extend this result to **graded** rings and modules, and produce explicit finite bounds above which, at a given degree, the direct limit in (A.15) is stationary. Without further ado, we present their results.

A.3 Computing graded local cohomology

Proposition A.3. *There exist functions $f_i \colon \mathbf{D} \longrightarrow \mathbf{N}^*$, for all $i \geq 0$, such that*

$$\mathrm{Ext}^i_S(S/B^l, S)_\delta \longrightarrow \mathrm{H}^i_B(S)_\delta$$

is an isomorphism for every $\delta \in \mathbf{D}$ and $l \geq f_i(\delta)$. Moreover, for P a finitely generated $\mathbf{D}$-graded S-module, and $F_\bullet$ a free resolution of P, with

$$F_i = \bigoplus_{\alpha \in \mathbf{D}} S(-\alpha)^{\oplus \beta_{i,\alpha}}$$

the map

$$\mathrm{Ext}^i_S(S/B^l, P)_\delta \longrightarrow \mathrm{H}^i_B(P)_\delta$$

is an isomorphism if for every $j \geq 0$ and every α such that $\oplus \beta_{j,\alpha} \neq 0$ we have

- $l \geq f_i(\delta - \alpha)$, *if $j = 0$;*

- $l \geq \max\{f_{i+j-1}(\delta - \alpha), f_{i+j}(\delta - \alpha)\}$, *if $j > 0$.*

Translating this into sheaf cohomology is immediate:[28]

Theorem A.4. *There exist functions $f_i \colon \mathbf{D} \longrightarrow \mathbb{N}^*$ for all $i \geq 0$ such that:*

$$\mathrm{Ext}^{i+1}_S(S/B^l, S)_\delta \longrightarrow \mathrm{H}^i(\mathcal{O}_X(\delta))$$

is an isomorphism for every $\delta \in \mathbf{D}$ and $l \geq f_i(\delta)$. Moreover, for P a finitely generated graded S module, with $F_\bullet$ a minimal free resolution of P, and

$$F_i = \bigoplus_{\alpha \in \mathbf{D}} S(-\alpha)^{\oplus \beta_{i,\alpha}} \, ,$$

the map

$$\mathrm{Ext}^{i+1}_S(S/B^l, P)_\delta \longrightarrow H^i_{\mathrm{tot}}(\tilde{P})_\delta$$

is an isomorphism if for every $j \geq 0$ and every α such that $\oplus \beta_{j,\alpha} \neq 0$ we have

- $l \geq f_{i+1}(\delta - \alpha)$, *if $j = 0$;*

- $l \geq \max\{f_{i+j}(\delta - \alpha), f_{i+j+1}(\delta - \alpha)\}$, *if $j > 0$.*

This gives a very practical and efficient way to compute local cohomology, and, as outlined above, sheaf cohomology on toric varieties. "All" one needs to do is evaluate the $\mathrm{Ext}^i_A(A/I^n, M)$ groups and keep track of the grading. Although this is a formidable task in general for humans, machines are very good at it. We implemented such routines using the symbolic algebra program *Macaulay 2* written by Dan Grayson and Mike Stillman [56].[29]

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[28] Both the printed and the electronic versions of [17] have typos in this statement.

[29] See [41] for a review on how *Macaulay 2* can be used to solve all sorts of problems in algebraic geometry.

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